M. CHASLES' MEMOIRS

ON

CONES AND SPHERICAL CONICS,

WITH

NOTES AND AN APPENDIX,

BY THE

REV. CHARLES GRAVES, F.T.C.D.

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TWO GEOMETRICAL MEMOIRS

ON THE

GENERAL PROPERTIES

OF

CONES OF THE SECOND DEGREE

AND ON THE

SPHERICAL CONICS,

BY

Michel

M. CHASLES.

TRANSLATED FROM THE FRENCH, WITH NOTES AND ADDITIONS,

AND

AN APPENDIX

ON THE

APPLICATION OF ANALYSIS TO SPHERICAL GEOMETRY,

BY THE

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PREFACE.

The two Memoirs by M. Chasles, "On the Properties of Cones of the second Degree," and "On the Spherical Conics," of which a translation is now presented to English geometers, were printed in the sixth volume of the Transactions of the Royal Academy of Brussels. Whether they are considered merely as exercises of pure geometry, exhibiting its elegance and power in a remarkable degree, or as a rich and early contribution to the theory of spherical curves, they possess strong claims on the attention of mathematicians.

But, published as they were, they remained unseen by the greater number of our geometers, so that it appeared highly desirable to make them more generally known by means of a reprint or translation; more especially as there is no detached work in the English language treating of the same subject. Most readers would doubtless prefer to see M. Chasles' Memoirs reprinted in their original language; but the small additional trouble which the translation of them has cost, will not have been vainly incurred, if a few readers benefit by it who are not perfectly familiar with the French idiom.

The matter contained in the Notes and Additions by the Translator is for the most part original. Some theorems relating to the anharmonic function of four points and to the involution of six points, have been borrowed from the notes to M. Chasles' admirable Histoire de la Geometrie.

In the Appendix will be found the outline of a system of analytic geometry, intended to accomplish for the surface of the sphere what the method of rectilinear coordinates has already effected for the plane. The Author has barely traced this outline: any reader tolerably conversant with the common processes of algebraic geometry will be able to fill it up; and none will complain of the incompleteness of a sketch which comprises in five and twenty pages the leading principles of the algebraic geometry of the sphere, and their application to the spherical conics. Those who are anxious to pursue this subject farther, will find formulæ for the transformation of spherical coordinates, and a discussion of the general equation of the second degree, in a paper which the Author had the honour of laying before the Royal Irish Academy on the 28th of June, 1841.

The twelfth volume of the Transactions of the Royal Society of Edinburgh contains two elaborate papers on the use of spherical coordinates, by Mr. S. T. Davies. The method which he employs leads him necessarily to unsymmetrical results, just such as we should meet with, if, in the analytic geometry of the plane, we restricted ourselves to the use of polar coordinates; and, in fact, Mr. Davies' method ought to be called that of *spherical polar coordinates.* But it may often be used with advantage; a few of the formulæ most necessary in its application are therefore given in the thirteenth section of the Appendix.

After the Author had constructed the theory of spherical coordinates, which he now lays before the reader, his attention was directed, by a note attached to one of Mr. Davies' papers, to articles published by Professor Gudermann, of Cleves, in Crelle's Journal of Pure and Applied Mathematics. From them he learned, to his regret, he must own, that he had been anticipated in the choice of his coordinates by Professor Gudermann, who has employed them successfully in investigating general properties of the spherical conics. However, as the Author has not yet been able to procure the work on Analytic Spherics, written by the Professor, and referred to in the papers inserted in Crelle's Journal, he is ignorant as to whether he has been also anticipated in the use of the differential calculus in discussing curves represented by an equation between spherical coordinates, and in the invention of general formulæ for the transformation of such coordinates. These steps being made, the theory is complete, and nothing remains but to apply it.

The Author had intended to add a second Appendix, containing those theorems relative to the plane conic sections, which may be deduced from the properties of the spherical conics stated in his Notes and Additions, but he leaves this to the reader, who will find no difficulty in doing it.

The Members of the Board of Trinity College, Dublin, have contributed, with their wonted liberality towards the expense of publishing this work. It is intended for the use of undergraduate students in the University of Dublin; and, it is hoped, may be useful in directing their prevailing taste for pure geometry to interesting and worthy objects.

ERRATA.

Page 51, line 17, second col. *for* foci, *read* focus.
—— 56, —— 22, second col. *for* given, *read* assumed.
—— 98, last line but one, *for* p, *read* $\frac{1}{2}p$.
—— 111, line 10, *for* $\tan^3 \nu$, *read* $4 \tan^3 \nu$.

MEMOIR

ON

THE GENERAL PROPERTIES

OF

CONES OF THE SECOND DEGREE.

ON

THE GENERAL PROPERTIES

OF

CONES OF THE SECOND DEGREE.

In a preceding Memoir on the general properties of the surfaces of revolution of the second degree, inserted by order of the Royal Academy in the collection of its memoirs, we demonstrated various properties of the cones of the second degree, which presented themselves as immediate consequences of the principles which we had founded upon the polar transformations of these cones and of the conic sections.

We now propose to discuss, by a direct method, some of the general properties of the cones of the second degree. We might have continued to employ the theory of polar transformations; but there is a more simple mode of proceeding, altogether independent of this theory. This mode is purely geometrical, requiring only a knowledge of the most elementary properties of the circle.

The only properties of the cones of the second degree which we shall assume as known are the two following:

"In every cone of the second degree there are three rectangular conjugate axes. There are also two series of circular sections, situated in planes parallel to two fixed planes."

This second proposition, which we receive without demonstration, as it is to be found in the elementary treatises on surfaces of the second degree, is the basis of our entire work;* and the numerous theorems to which we shall be led will all be deductions from this principle; and easy deductions, inasmuch as they will appear to result from elementary properties of the circle; so that we might say that our various theorems

* This general property of the cones of the second degree is due, I believe, to Descartes, who demonstrated it by the new principles of his geometry. His proof is to be found in the 6th vol. of his *Lettres* (12mo. ed. 1724, 1725.)

are, as it were, inscribed upon the planes of the circular sections of the cones of the second degree.

These theorems might also be demonstrated by algebraic analysis; but this method, which in general offers so great advantages, loses them all in this case, since it often requires very tedious calculations, and exhibits no connexion between the different propositions; so that it is only useful in verifying those which are already known, or whose truth has been otherwise suggested as probable.

SECTION I.

PRELIMINARY CONSIDERATIONS.

1. A cone of the second degree is one which has a conic section for its base.

Let there be two tangent planes; their traces on the plane of the conic section will be two tangents to this curve; the plane containing the two sides of the cone along which it is touched by the tangent planes, will meet the plane of this conic section in the chord which joins the points of contact of the two tangents; this chord is the polar, with relation to the conic section, of the point of concourse of the two tangents: hence we say that the plane of the two sides, along which the two tangent planes touch the cone, is the *polar plane*, with relation to the cone, of the right line of intersection of these two tangent planes; and this right line is called the *polar* of the plane of these two sides of contact.

It is well known that, in the conic section which is the base of the cone, all the right lines drawn through the same point have their poles upon the polar of this point; it follows, therefore, that

In a cone of the second degree, all the planes passing through the same axis have their* polars *situated in the* polar plane *of this axis.*

Hence, if, in the polar plane of any axis, we draw arbitrarily a second axis, the polar plane of this latter will pass through the first axis; and so these two axes are called *conjugate:* their polar planes are also said to be mutually *conjugate;* so that *two diametral planes of a cone of the second degree are* conjugate, *when the polar of one of them is in the other plane.*

2. Since every transversal plane, which cuts the cone in a conic section, cuts the polar plane of any axis in a right line

* We apply the name "axis" to every right line passing through the vertex of the cone.

which is the polar, with relation to the conic section, of the point where this plane meets this axis, it follows that if this transversal plane be parallel to the polar plane of the axis, it will cut the cone in a conic section whose centre will be the point where it cuts this axis, since the polar of this point will be at an infinite distance.

Hence, *the polar plane of any axis of a cone is such that every plane which is parallel to it cuts the cone in a conic section whose centre is upon this axis;* and conversely, *the polar axis of a plane is the geometrical locus of the centres of the sections made in the cone by planes parallel to this plane.*

3. Through the vertex of a cone of the second degree, we might draw, in an infinite number of different ways, three right lines such that each of them should be the polar of the plane of the two others. For, after having taken the polar plane of one right line, let us take the polar plane of a second right line drawn in this first polar plane; these two planes will intersect in a third right line, which will be the polar of the plane of the two first; so that these three right lines will be so related to each other that each of them will be the polar of the plane of the two others; and consequently, each of them is the locus of the centres of the sections made in the cone by planes parallel to the plane of the two others: these three right lines form a *system of conjugate axes.*

Hence, *in every cone of the second degree there is an infinite number of systems of three conjugate axes.*

4. It is clear that three conjugate axes meet a transversal plane, in three points of which each is the pole of the right line which joins the two others, with relation to the conic section in which this plane cuts the cone; if, therefore, this plane be parallel to the plane of two of the three conjugate axes, it will cut the two other faces of the trihedral angle formed by these three axes, along two conjugate diameters of the conic section in which this plane cuts the cone.

These properties of three conjugate axes of a cone of the second degree shew that it is very useful in analytic geometry to take three conjugate axes for the three axes of coordinates, x, y, z, because then the equation of the cone contains only the squares of the three coordinates. For every plane parallel to the plane of two of these axes will cut the cone in a conic section which will have its centre upon the third axis, and two conjugate diameters respectively parallel to the two first axes; consequently, the equation of the projection of this conic section upon the plane of the two first axes will be referred to two conjugate diameters, and will, therefore, contain only the squares of the two coordinates; which proves that the equation of the cone itself contains only the squares of the three coordinates.

5. It is known that, amongst all the systems of conjugate axes belonging to a cone of the second degree, there is one in which the three axes are rectangular. Hence, every plane perpendicular to one of these axes cuts the cone in a conic section which has its centre upon this axis, and its principal diameters parallel to the two other axes.

One of these axes lies in the interior of the cone, and every plane which is perpendicular to it cuts the cone in an ellipse; the two other axes are outside the cone, and every plane perpendicular to either of them cuts the cone in a hyperbola whose asymptotes are parallel to the two sides of the cone, which are parallel to the cutting plane.

Of the three rectangular conjugate axes, the one which lies in the interior of the cone will henceforward be designated as the *principal axis of the cone;* that which is parallel to the major axis of the ellipse, in which the cone is cut by a plane perpendicular to the principal axis, will be called the *major axis* of the cone; and the one which is parallel to the minor axis of this ellipse, will be denominated the *minor axis* of the cone.

The plane which contains the principal axis and the major axis will be the plane of the *greatest section* of the cone; the plane which contains the principal axis and the minor axis will be the plane of the *least section* of the cone; and finally, the plane which contains the major and the minor axes will be the *principal plane* of the cone.

It is manifest, that the principal plane does not cut the cone along any side; that, among all planes which can be drawn through the principal axis, the plane of the greatest section is that which cuts the cone along the two sides which contain between them the greatest angle; and that the plane of the least section is that which cuts it along the two sides which contain between them the least angle.

The tangent planes to the cone passing through the axis major, touch it along the two sides contained in the plane of the least section; the tangent planes passing through the minor axis touch it along the two sides contained in the plane of the greatest section; and finally, through the principal axis no tangent plane can be drawn to the cone.

It is further to be remarked, that every plane passing through the principal axis cuts the cone along two sides, which make equal angles with this axis; and that if a plane, passing through the major or minor axis, cuts the cone, the two sides of intersection will also make equal angles with this axis.

If the section of the cone, made by a plane perpendicular to its principal axis, be a circle, instead of being an ellipse, all the sides of the cone will make the same angle with this principal axis, and the cone will be one of revolution.

6. Let there be a cone of the second degree; through its vertex let us draw right lines perpendicular to its tangent planes, they will form a second cone, which will be of the second degree; since every plane passing through its vertex can only cut it along two sides; for if a plane were to cut it along three sides, these sides would be perpendicular to three planes touching the proposed cone, which planes would pass through the same right line perpendicular to the plane of the three sides; which is impossible, since through a right line only two tangent planes can be drawn to a cone of the second degree. Consequently, the second cone is of the second degree.

Two adjacent sides of this cone, that is, two sides infinitely near, are perpendicular to two adjacent planes touching the proposed cone; the plane of these two sides is perpendicular to the right line of intersection of the two adjacent tangent planes; that is to say, the tangent planes to the second cone are perpendicular to the sides of the first; we have, therefore, the following theorem:

The right lines drawn through a fixed point perpendicular to the tangent planes to a cone of the second degree, form a second cone of the second degree;

And the tangent planes to this second cone are perpendicular to the sides of the first.

7. Hence, to each side A in the first of the two cones *corresponds* a plane P touching the second cone, which plane is perpendicular to this side A.

This plane P touches the second cone exactly along the side which corresponds to the plane touching the first cone along the side A. For the side of contact of the plane P and of the second cone is perpendicular to a plane touching the first cone; this plane will necessarily pass through the perpendicular to the plane P, which is the side A of the first cone; it will therefore touch this cone along this side A. Therefore,

To a side of the first cone and to the tangent plane passing through this side, correspond a tangent plane to the second cone and the side of contact with this tangent plane.

8. To two planes touching the first cone correspond two sides of the second cone; the plane of these two sides is perpen-

dicular to the right line of intersection of the two tangent planes, and the planes touching the second cone along these sides are perpendicular to the sides of contact of the two planes touching the first cone; their right line of intersection is perpendicular to the plane of these two sides of contact; therefore,

To a right line and to its polar plane, with relation to the first cone, correspond a plane and its polar, with relation to the second cone.

9. Hence it follows, that *to two conjugate axes of the first cone correspond two conjugate planes of the second cone.*

For the polars of these two planes, with relation to the second cone, will correspond to the two polar planes of the two axes of the first cone; these two planes mutually pass through these two axes, since these axes are conjugate; therefore, the two right lines which correspond to these planes are mutually in the diametral planes of the second cone, which proves that these two planes are conjugate (1).

10. To two right lines drawn arbitrarily through the vertex of the first cone, will correspond in the second cone two planes, containing between them an angle equal to the supplement of that of the two right lines; the intersection of these two planes will be perpendicular to the plane of the two right lines.

These relations between the two cones are the same as those which exist between two supplementary trihedral angles, or between two supplementary spherical triangles; on this account we shall call the two cones *supplementary* one to the other.

From what has been said, it is plain that the properties relative to the angles contained by certain planes and right lines passing through the vertex of the first cone will give rise to properties of corresponding right lines and planes in the second cone; so that *the properties of cones of the second degree are double*, as well as those of spherical triangles.

11. Let us suppose that the two supplementary cones have the same vertex.

To three conjugate diameters of the first cone will evidently correspond three conjugate diametral planes of the second cone (9). Therefore, to the three rectangular conjugate diameters of the first cone will correspond the three rectangular conjugate diametral planes of the second. To the principal axis of the first cone will correspond the principal plane of the second; so that the two cones will have the same principal axis and the same principal plane. But the plane of the greatest section of the first will be the plane of the least section of the second, since the angle between the two sides of the second cone contained in this plane will be the supplement of the angle between the two sides of the first cone contained in this plane: in like manner, the plane of the least section of the first cone will be the plane of the greatest section of the second. It follows, that the major axis of the first cone will be the minor axis of the second, and the minor axis of the first cone will be the major axis of the second.

12. It is known that every cone of the second degree may be cut in circular sections by two series of planes parallel to two fixed planes; and that these two fixed planes being drawn through the vertex of the cone, pass through its major axis, and are therefore perpendicular to the plane of the least section.

As these two fixed planes possess a great number of properties which we are about to state, we are under the necessity of designating them by a particular name, and we shall call them *cyclic planes*.

Thus, the *cyclic planes* of a cone of the second degree are two fixed planes passing through its vertex, and parallel to the planes of the circular sections of the cone.

13. Two conjugate axes of the cone contained in a cyclic plane, are always at right angles, since they are parallel to two conjugate diameters of the section of the cone made by a plane parallel to the cyclic plane, this section being a circle; hence,

The cyclic planes of a cone possess this characteristic property, that

Two conjugate axes of the cone contained in a cyclic plane are always at right angles.

14. To the cyclic planes of a cone correspond, in the supplementary cone, two right lines perpendicular to these cyclic planes. Two conjugate axes of the first cone, contained in a cyclic plane, being at right angles (13), we infer that two conjugate planes of the supplementary cone, passing through one of the two right lines just mentioned, are at right angles.

A plane perpendicular to one of these right lines will cut the cone in a conic section, and the two conjugate planes in two right lines perpendicular to each other, and such that the pole of one of them, with relation to the conic section, will be upon the other right line (1). Thus, the point where the cutting plane meets the right line in question is such that every secant passing through this point has its pole, with relation to the conic section, upon the perpendicular to this secant passing through the point. This proves that the point *is a focus of the conic section.**

Hence, every plane perpendicular to one of the two right lines just mentioned cuts the cone in a conic section, one of whose foci is upon this right line.

These two right lines, being perpendicular to the two cyclic planes of the first cone, are in the plane of its least section (12), and are consequently in the plane of the greatest section of the supplementary cone (11). We have, therefore, this general property of cones:

In every cone of the second degree there are two right lines lying in the plane of its greatest section, which possess the property that every plane perpendicular to one of them cuts the cone in a conic section, one of whose foci is upon this right line.

In the first memoir above referred to, we proved the existence of these two right lines by means of the theory of polar transformations, and we called them *focal lines* in consequence of the characteristic property which we have just stated; but we did not there observe that *two conjugate planes passing through one of these two right lines are always at right angles.*

15. From what precedes it results, that

In two cones of the second degree, supplementary one to the other, the focal lines of one correspond to the cyclic planes of the other.

This theorem is very important, since it follows from it that the properties of the focal lines are consequences from those of the cyclic planes, and *vice versâ.*

* In what follows, we shall not take for granted any of the properties of the foci of the conic sections; these properties might even be deduced from those which we are about to demonstrate with reference to the cones of the second degree. However, it is necessary to define the foci of a conic section by some one of their properties. That which we have employed here to characterize these points is not, it is true, the most generally known; but it is known, and we are indebted for it to De Lahire; M. Poncelet has proved it, and has applied it with advantage in his "Traité des Propriétés Projectives" (p. 260, Nos. 451 and 453.) We shall take another opportunity of showing that this property adapts itself to the study of the conic sections with more readiness than many others equally characteristic. Moreover, it must be remarked, that we here define the foci of a conic section by the property in question merely to justify the designation of *focal lines,* which we adopt for the two axes of a cone of the second degree perpendicular to the cyclic planes of the supplementary cone. But all the properties of these two axes which we are about to prove are quite independent of the foci of the conic sections; and any one of these properties might as well be fixed upon to justify the designation of focal lines.

Thus, it will be sufficient for us to prove the one, and merely to state the others, without actually giving a particular demonstration of these latter.

16. First, let us lay down some properties of the circular sections of a cone, which we shall have to make constant use of.

When we consider two circular sections of a cone whose planes are not parallel, we call them *antiparallel* or *subcontrary* sections.

Through each point on the surface of a cone of the second degree, two subcontrary sections can be made to pass; for it is sufficient to draw through this point two planes parallel to the two cyclic planes of the cone.

17. *Every sphere passing through a circular section of a cone cuts this cone in a second subcontrary circular section.*

For, any side of the cone penetrates the sphere in two points, the rectangle under the distances of which from the vertex of the cone is constant: one of these points is on the plane of the circle through which the sphere has been made to pass; consequently, the other point is upon a sphere passing through the vertex of the cone, and having its centre upon the perpendicular let fall from the vertex upon the plane of this circle; (see the note at the end of this memoir;) the intersection of the first sphere and the cone is upon this second sphere, which proves that this intersection is a circle. The plane of this second circle cannot be parallel to the plane of the first; for in that case the cone would evidently be one of revolution. This second circle is therefore a subcontrary section with relation to the first: the theorem is therefore demonstrated.

It follows from hence, that

A sphere can be made to pass through any two subcontrary sections of a cone of the second degree.

18. Let us suppose that the plane of one of the two circles approaches indefinitely towards the vertex of the cone; as this plane is always parallel to a cyclic plane, when this circle degenerates into a point which is the vertex of the cone, the sphere will touch this cyclic plane; therefore,

Every sphere passing through a circular section and through the vertex of a cone of the second degree, touches a cyclic plane;

And conversely,

Every sphere passing through the vertex of a cone of the second degree, and touching a cyclic plane, cuts the cone in a circle, whose plane is parallel to the second cyclic plane of the cone.

SECTION II.

PROPERTIES OF THE TWO CYCLIC PLANES OF A CONE CONSIDERED SIMULTANEOUSLY; AND PROPERTIES OF THE TWO FOCAL LINES OF A CONE CONSIDERED SIMULTANEOUSLY.

19. The properties of the cyclic planes of a cone are of two kinds; some have reference to these two planes considered simultaneously, others to only one of these planes. It is as easy to state the one as the other; but as the former appear in some degree to be more characteristic, we shall commence with them.

The two focal lines of a cone, as appears from theorem (15), will be found to possess properties corresponding to those of the two cyclic planes.

20. Let there be a cone of the second degree; let it be cut by two planes parallel to its two cyclic planes; the sections will be two circles lying on the same sphere (17). Each side of the cone meets these circles, and the tangent plane to the cone along this side cuts their planes in two right lines which are the tangents to these circles, passing through the points where the side meets them: these two right lines are therefore tangents to the sphere which passes through the two circles. Now, two tangents to a sphere lying in the same plane make equal angles with the chord which joins the two points of contact; here this chord is the side of the cone; the two tangents are respectively parallel to the two right lines in which the tangent plane intersects the two cyclic planes. We have, therefore, the following property of the cyclic planes, and, consequently, the property of the focal lines stated in the second column:

Every tangent plane to a cone of the second degree intersects the two cyclic planes in two right lines, which make equal angles with the side of the cone along which it is touched by the tangent plane.

The planes passing through the two focal lines of a cone of the second degree and through any side whatever, make equal angles with the plane touching the cone along that side(*a*).

(*a*) This theorem and the second in No. 24, have been already given by M. Magnus, of Berlin (Annales de Mathématiques, August, 1825.) As to all the other properties of the cones of the second degree contained in this memoir, we believe they are quite new, with the exception of some which we have already stated in our Memoir on the surfaces of revolution of the second degree, inserted in the 5th vol. of the New Memoirs of the Royal Academy of Brussels.

21. Conversely,

If a cone be such that its tangent plane, passing through any side, intersects two fixed planes in two right lines, making equal angles with that side, this cone is of the second degree.

If a cone be such that the planes passing through two fixed axes and through any side whatever, make equal angles with the plane touching the cone along that side, this cone is of the second degree.

In order to demonstrate the first theorem, let us draw two planes P, P′, parallel to the two fixed planes. Let M, M′, M″, M‴, be the points where four consecutive sides infinitely near to one another meet the first plane P, and let m, m', m'', m''', be the points where these same sides meet the second plane P′.

Let us imagine the circle which passes through the three points M, M′, M″, to be the base of a second cone, having the same vertex S, as the proposed cone; the section of this cone made by the second plane P′ will pass through the points m, m', m'', and this curve will be a circle; for (by hypothesis) the two right lines MM′, mm', making equal angles with the side SMm, the plane of the circular section of the second cone passing through the point m will pass through the right line mm', as appears from the preceding theorem; the two tangents M′M″, $m'm''$ also making equal angles with the side SM′m', the plane of this circular section will likewise pass through the tangent $m'm''$; this plane will, therefore, be exactly the plane of the three points m, m', m''; so that P, P′ will be the planes of two subcontrary sections of the second cone: the tangents to these two sections passing through the points M″, m'', will, therefore, make equal angles with the side SM″m'' (20); but the two right lines M″M‴, $m''m'''$ make (by hypothesis) equal angles with that side; these two right lines are, therefore, actually the tangents to the two circular sections of the second cone: for, through the two points M″, m'' only two right lines can be drawn in the planes P, P′, which make equal angles with the side SM″m'', since these two right lines intersect at the point where the plane, drawn at right angles to this side through the middle point of the segment M″m'', meets the intersection of the two planes P, P′.

Hence, it is proved that the circle passing through three points infinitely near, M, M′, M″, assumed in the section of the proposed cone made by a plane parallel to one of the two fixed planes, passes always through a fourth point M‴ of this curve infinitely near the former; for the same reason, this circle, passing through the three consecutive points M′, M″, M‴, will pass through a fifth point, and again, through a sixth; which proves that this base is itself a circle; or, in other words, the osculating circle of this curve has at every point in it a contact with it of the third order, from which it follows that this curve can only be a circle. Thus we have proved the first theorem, and consequently the second.

22. The two theorems (20) are particular cases of the following:

Every plane passing through two sides of a cone of the second degree intersects the cyclic planes in two right lines which respectively make equal angles with these two sides.

The planes passing through the two focal lines of a cone of the second degree and through the right line of intersection of two tangent planes to the cone respectively make equal angles with these two tangent planes.

It is sufficient to prove the first of these two theorems.

For this purpose, let us take two subcontrary sections of the cone; the plane of the two sides cuts the planes of these two circles along two chords, which form, with the portions of the two sides lying between these chords, a plane quadrilateral, inscribed in the circle in which the plane of this quadrilateral intersects the sphere, on the surface of which are the two subcontrary sections; two opposite angles of this quadrilateral are, therefore, supplemental one to the other; hence the two chords respectively make equal angles with the two sides of the cone. These chords are parallel to the right lines in which the plane of the two sides intersects the two cyclic planes; the first theorem is, therefore, proved; the second follows from it.

In the memoir above referred to we had already proved these two theorems in two different ways; first, as a consequence from the properties of the lines of curvature of a hyperboloid of one sheet; and again, as a consequence from the properties of surfaces of revolution of the second degree.

23. Two planes touching a cone of the second degree along any two sides, intersect the two cyclic planes in four right lines, which are the generatrices of the same cone of revolution, whose axis of revolution is perpendicular to the plane of the two sides of contact.

The four vector planes, passing through the two focal lines of a cone of the second degree and through any two sides of the cone, are tangents to the same cone of revolution, whose axis of revolution is the right line of intersection of the two planes touching the proposed cone along the two sides.

Let us prove the first theorem.

The first tangent plane intersects the two cyclic planes in two right lines, making equal angles with the side of contact (20); therefore, these two right lines make equal angles with every plane passing through this side; consequently, they make equal angles with the plane of the two sides. In like manner the second tangent plane intersects the two cyclic planes in two right lines, which make equal angles with this same plane of the two sides.

Now let us imagine a plane parallel to one of the cyclic planes; it will cut the cone in a circle, and the two tangent planes to the cone in two tangents to this circle; these two tangents will be parallel to the two right lines in which the cyclic plane intersects the two tangent planes; but these two tangents make equal angles with the chord which joins their points of contact, and, consequently, they make equal angles with every plane passing through this right line; they, therefore, make equal angles with the plane of the two sides of contact with the tangent planes.

Hence, it follows, that the four right lines of intersection of the two cyclic planes with the two tangent planes, make equal angles with the plane of the two sides; consequently, they make equal angles with the right line perpendicular to this plane, which proves that they are generatrices of the same right cone, whose axis of revolution is perpendicular to the plane of the two sides of contact. Q. E. D.

24. The sum or the difference of the angles, which each tangent plane to a cone of the second degree makes with the two cyclic planes, is constant.	The sum or the difference of the angles, which each side of a cone of the second degree makes with its two focal lines, is constant. (See note to 20.)

These two theorems may be respectively deduced from the two preceding ones in the same manner: we mean to prove the second, since the figure it requires is easily constructed.

Let there be a sphere, whose centre is at the vertex of a cone, it will meet the two focal lines, (supposed to be prolonged within a single sheet of the cone,) in two points F, F′; it will meet any two sides of the cone in two points, *m*, *n*; and the four vector planes passing through these sides, in four arcs of great circles, which will touch a small circle of the sphere formed by the intersection of the sphere with the right cone, to which the four vector planes are tangents. (23, second column.)

As the arcs F*m*, F′*m*, F*n*, F′*n*, measure the angles which the two focal lines make with the two sides of the cone, our object is to prove that the sum of the first two is equal to the sum of the two latter.

Let *a*, *a*′, *b*, *b*′, be the points where these four arcs respectively touch the small circle.

The two arcs *ma*, *ma*′ are equal, as being drawn from the same point *m*, to touch the small circle; hence we conclude that $Fm + F'm = Fa + F'a'$.

In like manner, the arcs *nb*, *nb*′ are equal, and it follows that $Fn + F'n = Fb + F'b'$; now, the arcs Fa and Fb are equal, the arcs $F'a'$ and $F'b'$ are also equal; the two right hand members of the two equations are therefore equal, and, consequently, the first members are likewise equal to one another; which was to be proved.

The angles made by the two generatrices of the cone with a focal line, are the supplements of the angles made by these generatrices with the production of this focal line within the second sheet of the cone; so that the difference of the angles made by a side of the cone with one focal line and with the production of the other focal line, is also a constant quantity; it was for this reason that in the enunciation of the theorem we said, *the sum or the difference* of the angles made by each side with the two focal lines. Thus the theorem is proved.

25. It is known, that if, in a spherical triangle, one angle be invariable, and also the product of the trigonometric tangents of the halves of the sides containing it; the area of the triangle will remain constant, (Legendre's Geometry; Note on the Area of the Spherical Triangle.) From the first of the two preceding theorems we may, therefore, deduce the first, and, consequently, the second of the two following ones:

In every cone of the second degree, each tangent plane intersects the two cyclic planes in two right lines such that the product of the tangents of the semi-angles, which they make with the intersection of the two cyclic planes, is constant.

In every cone of the second degree, the vector planes, passing through the two focal lines and through any side of the cone, are such that the product of the tangents of the semi-angles, which they make with the plane of the two focal lines, is constant.

It would be easy to prove these two theorems directly, without making use of the proposition in spherical trigonometry to which we referred; but as this proposition is to be found in an elementary work familiar to all geometers, we may be allowed, in order to save time, to avail ourselves of it; moreover, we shall hereafter give the direct proof of which we speak, and which will exhibit the two theorems above stated, as depending upon the most elementary principles of geometry.

26. Let there be two subcontrary sections of a cone of the second degree; each side of the cone meets these two circles in two points, the rectangle under the distances of which from the vertex of the cone, is invariable, whatever be this side, since these two circles are upon the same sphere (17). If from the vertex of the cone, perpendiculars be let fall on the planes of the two circles, they will be respectively equal to these two distances, respectively multiplied by the sines of the angles, which the side of the cone makes with the planes of

the two circles; therefore, the rectangle under the two perpendiculars will be equal to the rectangle under the two distances, multiplied by the product of the two sines. Now, the perpendiculars will be the same, whatever side of the cone we consider; the rectangle under the two distances is also the same, as we have just shown; therefore, the product of the two sines is constant; but the planes of the two circles are parallel to the two cyclic planes of the cone; we have, therefore, the first of the two following theorems, and, consequently, the second.

In every cone of the second degree, the product of the sines of the angles, which each side makes with the two cyclic planes, is constant.

In every cone of the second degree, the product of the sines of the angles, which each tangent plane makes with the two focal lines, is constant.

27. If from a point o, taken arbitrarily, perpendiculars be let fall upon the tangent planes to a cone of the second degree, they will form a second cone of the second degree, which will be the supplementary one to the first (10). The feet of these perpendiculars will be upon the sphere whose diameter is the right line joining the point o with the vertex of the given cone, since the right line drawn from this vertex to the foot of each perpendicular, makes a right angle with that perpendicular. Hence the feet of the perpendiculars will be upon the curve of intersection of the sphere and the second cone.

Let us suppose the point o to be upon a focal line of the given cone. This right line is perpendicular to a cyclic plane of the second cone (15); the sphere whose centre is upon this right line will, therefore, touch this cyclic plane, which proves that it will intersect the second cone in a circle lying in a plane parallel to the second cyclic plane of this cone (18); this second cyclic plane is perpendicular to the second focal line of the given cone (15); we have, therefore, the following theorem:

If from a point assumed upon a focal line of a cone of the second degree, perpendiculars be let fall upon the tangent planes to this cone, their feet will be upon a circle, the plane of which will be perpendicular to the second focal line of the cone.

28. The right lines which join the vertex of the given cone with the feet of the perpendiculars are the orthogonal projections of the first focal line upon the tangent planes; these right lines form a cone having for its base the circle which is the locus of the feet of the perpendiculars. This cone is evidently symmetrical on each side of the plane of the greatest section of the given cone in which the focal line lies;

and it touches this given cone along the two sides lying in this plane, since these two sides are the projections of the focal line upon the tangent planes along those sides. From this position of the cone it results, that the orthogonal projections of the second focal line upon the tangent planes will be upon this same cone. This cone will therefore pass, as appears from the theorem which we have just proved, through a second circle lying in a plane perpendicular to the second focal line; whence we deduce the first of the two following theorems, and consequently the second:

The orthogonal projections of the two focal lines of a cone of the second degree upon the tangent planes to the cone form a second cone of the second degree, which has a double contact with the proposed cone, and whose cyclic planes are perpendicular to the focal lines of the latter.	If a right angle, having the same vertex with a cone of the second degree, turn round this vertex, so that one of its sides moves along either of the two cyclic planes of the cone, whilst its other side moves round the cone; the plane of this angle will envelope a cone of the second degree, which will have a double contact with the given cone, and whose focal lines will be perpendicular to the two cyclic planes of the latter.

29. Theorem (27) gives rise to the following, which is the converse of it:

If through the different points of a circle traced upon a cone of the second degree planes be drawn perpendicular to the sides passing through these points, these planes will envelope a second cone of the second degree, one of whose focal lines will pass through the vertex of the given cone, and whose other focal line will be perpendicular to the plane of the circle.

30. If from a point in the focal line of a cone, perpendiculars be let fall upon two tangent planes, the right line joining their feet will be at right angles with the right line of intersection of the two tangent planes; now, by theorem (27), the former right line will also be at right angles with the second focal line of the cone; it is, therefore, perpendicular to the plane passing through the right line of intersection of the two tangent planes and through the second focal line; therefore,

If from a point in a focal line of a cone of the second degree, perpendiculars be let fall upon two tangent planes, the plane passing through the vertex of the cone and perpendicular to the right line joining their feet will pass through the second focal line of the cone.

31. This may be otherwise stated in the first of the two following theorems, from which the second is a consequence:

If a focal line of a cone of the second degree be orthogonally projected upon two tangent planes, the plane passing through the right line of intersection of the two tangent planes, and perpendicular to the plane containing the two projections, will pass through the second focal line.

If two right angles have each a side upon a cyclic plane of a cone of the second degree, and their two other sides coincident with two sides of the cone, the right line of intersection of their planes and the right line in which the plane of the two sides intersects the second cyclic plane will be at right angles.

32. Let there be two cones of the second degree, having the same vertex and the same cyclic planes; if a common tangent plane be drawn to them, it will touch the two cones along two sides, each of which will make equal angles with the two right lines in which this tangent plane intersects the two cyclic planes (20.) These two sides will therefore bisect both the angle and the supplement of the angle contained by these two right lines, which proves that these two sides are at right angles. We have, therefore, the two following theorems:

When two cones of the second degree have the same vertex and the same cyclic planes, if a common tangent plane be drawn to them, the two sides of contact lying in this plane will be at right angles to each other.

If two cones of the second degree, which have the same vertex and the same focal lines, intersect one another, their tangent planes passing through each side of intersection will be at right angles, that is to say, the two cones will cut one another at right angles.

M. Dupin and M. Binet, Jun. have stated the general conditions to be fulfilled in order that two surfaces of the second degree may cut every where at right angles; but they did not include in their researches the case of two conical surfaces, which is disposed of in the second of the two theorems just proved. (See Développemens de Géométrie de Ch. Dupin, and, 16[e] cahier des Journaux de l'école Polytechnique.)

These two theorems may be employed with great advantage in different investigations, as we shall hereafter have occasion to show.

SECTION III.

PROPERTIES OF A CONE OF THE SECOND DEGREE RELATING TO A SINGLE CYCLIC PLANE; AND PROPERTIES RELATING TO A SINGLE FOCAL LINE.

33. The properties of a cone of the second degree relating to a single cyclic plane, which we are about to prove, are deduced immediately from the properties of the circle; and those relative to a single focal line may be shewn to depend

upon the former by reference to the supplementary cone. Hence, we shall content ourselves with merely writing them alongside of the former.

The right line polar to a plane, with relation to a cone, passes, as we have stated (2,) through the centre of the section made in the cone by this plane or by any other parallel to it. Hence, the polar of a cyclic plane is the right line which is the locus of the centres of the circular sections of the cone lying in planes parallel to this cyclic plane.

We have seen that, to a cyclic plane and to its polar, correspond, in the supplementary cone, a focal line and its polar plane (14.) We shall call this polar plane the *director plane* of the cone, a name analogous to that of directrix in the conic sections. A cone has two director planes, but we must always be understood to refer to that which corresponds to the focal line which we are considering.

34. Let us begin with two theorems, the second of which is remarkable for its analogy to the leading property of the directrices of the conic sections.

In every cone of the second degree the ratio of the sines of the angles made by each tangent plane with a cyclic plane and with the polar of this cyclic plane is constant.

In every cone of the second degree the ratio of the sines of the angles which each side makes with a focal line and with the director plane is constant.

To prove the first of these two theorems, let us take a cone of the second degree and a secant plane parallel to one of its cyclic planes; the section of the cone will be circle, and the axis of the cone passing through the centre of this circle will be the polar of the cyclic plane. Let there be a tangent plane to the cone, and from the centre of the circle let us drop a perpendicular on this plane; the sine of the angle made by this tangent plane with the plane of the circle is equal to this perpendicular divided by the radius of the circle; the sine of the angle made by the tangent plane with the axis of the cone terminating in the centre of the circle is equal to this same perpendicular divided by the length of this axis. From the form of these expressions for the sines of the two angles, it appears that their ratio is independent of the perpendicular, and contains only the radius of the circle and the distance of its centre from the vertex of the cone. Hence, this ratio is constant, whatever be the tangent plane to the cone: which proves the first of the two theorems above stated; the second is deduced from it by reference to the supplementary cone.

35. In what follows we shall still consider, as we have just done, a circular section of the cone, and we shall regard the

points and right lines lying in the plane of this section as the intersections of this plane, with right lines and planes passing through the vertex of the cone; and, in order to apply to the cone the various properties of the circle, in place of the right lines lying in the plane of this circle, we shall substitute, in the statement of our theorems, the traces upon the cyclic plane of the planes passing through these right lines and through the vertex of the cone.

36. Thus, in order to transfer to the cone, this property of the circle, "a radius is perpendicular to the tangent passing through its extremity," we shall consider the tangent and the radius as being the traces of a tangent plane to the cone and of a plane passing through the side of contact and through the polar of the cyclic plane (since this polar passes through the centre of the circle;) and we shall remark that these two traces are respectively parallel to the traces of the same two planes upon the cyclic plane parallel to the plane of the circle; whence we have the first of the two following theorems, and, consequently, the second.

The tangent plane to a cone of the second degree, and the plane passing through the side of contact and through the polar of a cyclic plane, meet this cyclic plane in two right lines which are at right angles with each other.

If through a focal line of a cone of the second degree two vector planes be drawn, of which the first passes through any side of the cone, and the second through the right line in which the plane touching the cone along that side meets the director plane, these two vector planes will be at right angles.

37. Two tangents to a circle make equal angles with the chord which joins the two points of contact. Hence we infer, in accordance with what has been said (35,) that

Two tangent planes to a cone of the second degree and the plane of the two sides of contact intersect a cyclic plane in three right lines, the third of which bisects the angle between the first two.

If through a focal line of a cone of the second degree vector planes be drawn respectively passing through two sides of the cone and through the right line of intersection of the two planes touching the cone along these sides, the third vector plane will bisect the angle between the first two.

38. Two tangents to a circle make equal angles with the right line which joins their point of concourse with the centre of the circle. Hence we infer, that

Two tangent planes to a cone of the second degree, and the plane passing through their right line of intersection and through the polar

The vector planes passing through a focal line of a cone of the second degree and through two sides make equal angles with the vector plane of a cyclic plane, meet that cyclic plane in three right lines, the third of which bisects the angle between the first two.

passing through the right line in which the plane of the two sides intersects the director plane.

39. The right line drawn from the centre of a circle to the point of concourse of two tangents is perpendicular to the chord which joins the two points of contact, and passes through the middle of this chord. Let it be observed, that the middle point of a right line is the harmonic conjugate to a point in this line lying at an infinite distance, and that the four right lines drawn from any point whatsoever to four harmonic points form a harmonic pencil; and we shall have the first of the two following theorems, and, consequently, the second.

The plane passing through the polar of a cyclic plane of a cone of the second degree and through the right line of intersection of two tangent planes to the cone, and the plane of the two sides of contact, meet the cyclic plane in two right lines which are at right angles with each other;

And the right lines, in which the the plane of the two sides meets the cyclic plane and the plane passing through its polar and through the right line of intersection of the two tangent planes, are harmonic conjugates with relation to the two sides.

Two vector planes passing through a focal line of a cone of the second degree will be at right angles, if one passes through the right line of intersection of two tangent planes to the cone, and the other through the right line in which the plane of the two sides of contact meets the director plane;

And the two planes drawn through the right line of intersection of the two tangent planes, and passing, the first through the focal line, and the other through the right line in which the plane of the two sides meets the director plane, are harmonic conjugates with relation to the two tangent planes.

40. *If the plane of the two sides passes through the polar of the cyclic plane, the right line of intersection of the two tangent planes will be in this cyclic plane; therefore,*

If through a right line lying in the cyclic plane of a cone of the second degree two tangent planes be drawn to the cone, the plane of the two sides of contact will intersect the cyclic plane in a second right line which will be at right angles to the former.

If the right line of intersection of the two tangent planes be in the director plane, the plane of the two sides of contact will pass through the focal line; therefore,

If a cone of the second degree be cut by a transversal plane passing through a focal line, the planes touching the cone along the two sides lying in this plane will intersect on the director plane; and the plane passing through the focal line and through the right line of intersection of the two tangent planes will be perpendicular to the transversal plane.

41. When a quadrilateral is inscribed in a circle two opposite angles are always supplemental one to the other. Hence we infer, that

When a tetrahedral angle is inscribed in a cone of the second degree, the angle between the traces of two of its adjacent faces upon a cyclic plane of the cone is supplemental to the angle contained by the traces upon the same cyclic plane of the two other faces.

When a tetrahedral angle is circumscribed about a cone of the second degree, the vector planes, passing through a focal line and through two adjacent edges of the tetrahedral angle, contain an angle which is supplemental to the angle contained by the vector planes passing through the two other edges.

42. All the angles whose vertices are upon the circumference of a circle, and whose sides pass through the extremities of the same chord, are equal (we mean the acute angle contained between the two sides, otherwise two angles would be supplemental one to the other, when their vertices were on opposite sides of the chord;) if this chord pass through the centre of the circle all the angles are right. Hence we infer, that

If through two fixed sides of a cone of the second degree, two planes be drawn, intersecting along any third side of the cone, the traces of these planes upon a cyclic plane will contain between them an angle of invariable magnitude.

Two fixed tangent planes being drawn to a cone of the second degree, and also any third tangent plane whatever, the vector planes passing through a focal line and through the two right lines in which the two fixed planes are intersected by the third tangent plane will contain between them an angle of invariable magnitude, whatever be this third tangent plane.

This angle will be right if the plane of the two fixed sides passes through the polar of the cyclic plane.

This angle will be right if the right line of intersection of the two fixed tangent planes be in the director plane corresponding to the focal line.

43. *The polar of a cyclic plane is in the plane of the least section of the cone. Hence we infer from the second part of the preceding theorem, that*

The director plane passes through the minor axis of the cone; the two planes touching the cone along the sides lying in the plane of the greatest section also pass through the minor axis; they therefore intersect upon the director plane. Hence we infer from the second part of the preceding theorem, that

If through the two sides of a cone of the second degree lying in the plane of its least section two planes be drawn intersecting along any side whatever of the cone, their traces upon one of the cyclic planes will be at right angles.

Every tangent plane to a cone of the second degree intersects the two tangent planes that are perpendicular to the plane of the greatest section in two right lines such that the vector planes, passing through a focal line and through these two right lines, are at right angles to each other.

44. From the two theorems (42,) the following immediately result:

If in a cyclic plane of a cone of the second degree, an angle of invariable magnitude, and having the same vertex with the cone, be made to turn; the plane passing through one side of the angle, and through a fixed side of the cone, will pass through a second side; and the plane determined by this second side of the cone and by the second side of the moveable angle will turn round a fixed side of the cone.

If round a focal line of a cone of the second degree, as an edge, a dihedral angle of invariable magnitude be made to turn; and if through the right line in which one of its faces meets a fixed tangent plane to the cone a second tangent plane to the cone be drawn, this second plane will meet the second face of the dihedral angle in a right line which will always be in the same tangent plane to the cone.

45. If, from the vertex of a cone, a perpendicular be let fall upon the plane of a circular section of the cone, and two parallel tangents be drawn to this circle, the sum of the distances of the foot of the perpendicular from the two tangents, will be constant and equal to the diameter of the circle; but, these distances may be taken as the trigonometric tangents of the angles, which the two tangent planes to the cone, passing through the two tangents to the circle, make with the perdicular let fall from the vertex of the cone, and these trigonometric tangents are equal to the reciprocals of the trigonometric tangents of the angles which the two tangent planes make with the plane of the circle, or with the cyclic plane to which the plane of the circle is parallel; hence we infer, that

If, through a right line lying in a cyclic plane of a cone of the second degree, two tangent planes be drawn to the cone, the sum of the reciprocals of the trigonometric tangents of the angles which they make with the cyclic plane will be constant.

Every plane passing through a focal line of a cone of the second degree cuts the cone along two sides such that the sum of the reciprocals of the trigonometric tangents of the angles which they make with this focal line is constant.

SECTION IV.

GEOMETRIC LOCI RELATING TO THE CYCLIC PLANES, AND TO THE FOCAL LINES OF CONES OF THE SECOND DEGREE.

46. If the sides of an angle of invariable magnitude, contained in a given plane, turn round two fixed points, its vertex generates an arc of a circle; hence we infer, that

If round two fixed right lines which intersect, two planes be made

Being given two fixed planes and a right line passing through a point to turn, so that their traces upon a fixed plane passing through the point of intersection of the two right lines may contain an angle of invariable magnitude, the intersection of the two moveable planes will generate a cone of the second degree, which will have the fixed plane for one of its cyclic planes, and which will pass through the two fixed right lines.

in their intersection, if round this right line two planes be made to turn, containing between them an angle of invariable magnitude, the two right lines in which these planes will respectively intersect the two fixed planes will determine a moveable plane, which will envelope a cone of the second degree, in which the fixed right line will be a focal line, and which will touch the two fixed planes.

It is scarcely necessary to mention, that, as upon a given right line, two arcs of circles can be described capable of containing the same angle; so, in each of these two propositions, there are two cones perfectly equal.

We might have deduced these two propositions directly from the two theorems (42.)

47. The vertex of an angle of invariable magnitude, whose sides touch a circle, generates a second circle; and the chord which joins the point of contact of the two sides, envelopes a third circle; these three circles are concentric; whence we infer, that

If two tangent planes to a cone of the second degree move in such a way that their traces upon a cyclic plane contain between them an angle of invariable magnitude, the intersection of these two planes will generate a second cone of the second degree.

The plane of the two sides of contact of the two tangent planes will envelope a third cone of the second degree.

The cyclic plane in question will be a cyclic plane of the two new cones, and this plane will have the same polar in the three cones.

If a dihedral angle of invariable magnitude turn round a focal line of a cone of the second degree, as an edge, the plane of the two sides along which its faces will meet the cone will envelope a second cone of the second degree.

The planes touching the given cone along these two sides will intersect upon a third cone of the second degree.

The focal line, round which the dihedral angle turns, will be a focal line of the two new cones; and the corresponding director plane will be the same in the three cones.

48. If an angle of invariable magnitude turn round its vertex, which lies upon the circumference of a circle, the chord intercepted between its sides will always be a tangent to another circle concentric with the former; therefore,

If round a side of a cone of the second degree two planes be made to turn, whose traces upon a cyclic plane contain between them an angle of invariable magnitude, the

If round a focal line of a cone of the second degree a dihedral angle of invariable magnitude be made to turn, and if through the right lines in which the faces of this angle meet plane of the two sides along which these two planes meet the cone will envelope a second cone of the second degree, which will have the same cyclic plane with the given one; and this plane will have the same polar in the two cones.

a fixed plane touching the cone two other tangent planes to the cone be drawn, their right line of intersection will generate a second cone of the second degree, in which the edge of the dihedral angle will be a focal line, and the corresponding director plane will be the same in the two cones.

49. Let us take a cone of the second degree, and cut it by any plane whatsoever, the curve of intersection will be a conic section; and it is known, that the vertices of all the right angles, circumscribed about this curve, are upon a circle which has the same centre with it; whence we infer, that

If two tangent planes be drawn to a cone of the second degree, so that their traces upon a fixed plane may contain between them a right angle, the intersection of these two planes will generate a second cone of the second degree, one of whose cyclic planes will be parallel to the fixed plane.

This plane will have the same polar in the two cones.

If two rectangular planes turn round a fixed right line passing through the vertex of a cone of the second degree, the sides along which these two planes cut the cone, taken two by two, will determine four planes which will envelope a second cone of the second degree, in which the fixed right line will be a focal line.

This right line will have the same polar plane with relation to the two cones.

50. If the fixed plane, in the former of these two theorems, be one of the three rectangular conjugate planes of the given cone, the second cone will evidently be one of revolution round the axis perpendicular to this plane; therefore,

If two tangent planes be made to revolve round a cone of the second degree, so that their traces upon one of the three rectangular conjugate planes of the cone shall always be at right angles, the intersection of these two planes will generate a cone of revolution round the axis perpendicular to that plane.

If round one of the three rectangular conjugate axes of a cone of the second degree two rectangular planes be made to turn, the sides along which these two planes will intersect the cone, taken two by two, will determine four planes which will envelope a cone of revolution round that axis.

51. It is known, that if the vertex of an angle of invariable magnitude traverse a right line, whilst one of its sides turns round a fixed point, its other side envelopes a parabola touching the right line traversed by the vertex of the angle. This proposition will help us in the proof of the following theorems:

If the faces of a variable dihedral angle inscribed in a cone of the second degree turn round two fixed sides of the cone, the plane determined by the two right lines in which these faces intersect the two cyclic planes of the cone will envelope a second cone of the second degree touching these two cyclic planes.

Two fixed planes being drawn so as to touch a cone of the second degree, a third moveable tangent plane will intersect them in two right lines, and the planes respectively passing through these two right lines, and through the two focal lines of the cone, will intersect in a right line, which will generate a cone of the second degree passing through these two focal lines.

It is sufficient to prove the first of these two theorems, since the second may be deduced from it by reference to the supplementary cone.

Let us draw a transversal plane parallel to a cyclic plane of the given cone; this plane will intersect the cone in a circle, the second cyclic plane in a right line D, and the faces of the dihedral angle in two right lines, L, L′, which will pass through two fixed points in the circle, and will intersect in any third point of this circle; the plane, of which we seek the enveloping surface, will be cut by the transversal plane in a right line which will pass through the two points in which the two right lines, L, L′, respectively meet the traces of the two cyclic planes upon the transversal plane; one of these traces is the right line D, the other is at an infinite distance. If, therefore, through the point in which the right line L meets the right line D, we draw a right line parallel to the right line L′, this parallel will be the trace of the plane whose enveloping surface we are seeking. Now, this trace envelopes a parabola; for it makes with the right line L an angle which is of invariable magnitude, as being equal to the angle between the two right lines, L, L′; the vertex of this angle moves along the fixed right line D; its side L turns round a fixed point in the circle; therefore, its second side envelopes a parabola touching the right line D; which proves that the moveable plane envelopes a cone of the second degree touching the cyclic plane, which intersects the transversal plane in the right line D. In like manner it may be proved that this cone will touch the second cyclic plane; the theorem is therefore proved.

52. If, round two fixed right lines which meet another, two rectangular planes be made to turn, their intersection will generate a cone of the second degree, which will pass through the two fixed right lines, and whose cyclic planes will be perpendicular to these two right lines.

If, round a point assumed in the intersection of two fixed planes, a right angle be made to turn, whose sides move in the two fixed planes, the plane of this angle will envelope a cone of the second degree, which will touch the two fixed planes, and whose focal lines will be perpendicular to these two planes.

For, when two planes are rectangular, every plane perpendicular to one of them intersects them in two right lines which are at right angles to each other; therefore, a transversal plane perpendicular to one of the two fixed right lines intersects the two moveable planes in two right lines which are at right angles to each other; now, these two right lines pass through the two fixed points in which the transversal plane meets the two fixed right lines; their point of intersection will, therefore, generate a circle which passes through these two points, and along which moves the right line of intersection of the two moveable planes; which proves the theorem.

For, let us draw through the vertex of the moveable angle a right line perpendicular to one of the two fixed planes; it will be perpendicular to the side of the angle which moves in this plane; the plane determined by this right line and by the other side, will, therefore, be perpendicular to the former side; whence it follows, that the planes passing through the right line and through the two sides of the angle will be rectangular. Now, we have seen that, when two rectangular planes turn round a fixed right line, the two right lines, in which they intersect two fixed planes passing through a point in this right line, are in a plane which envelopes a cone of the second degree, in which the fixed right line is a focal line (46, *second column;*) *which proves the theorem.*

It would have been sufficient to prove one of these two theorems, since either might be deduced from the other by reference to the supplementary cone.

53. If, round a fixed point, a right line be made to turn, which makes, with two fixed planes, angles, the product of whose sines is constant, this right line will generate a cone of the second degree whose two cyclic planes will be parallel to the two given planes.

If, round a fixed point, a plane be made to turn, which makes, with two given right lines, angles, the product of whose sines is constant, this plane will envelope a cone of the second degree whose focal lines will be parallel to the two given right lines.

These two propositions are evidently the converses of the two theorems (26.)

It is sufficient to prove the first.

For this purpose, let us take O as the fixed point; M, m, as the points in which a side of the cone generated by the moving right line meets the two fixed planes; and P, p, as the feet of the perpendiculars let fall from the point O upon these planes; the sines of the angles which this side makes with these planes are respectively equal to $\frac{OP}{OM}$, $\frac{Op}{Om}$; the product of these two ratios ought therefore to be constant. Now, the numerators are constant, whatever side of the cone we consider; therefore, the rectangle under the two lines, OM, Om, is constant; which proves that the point M being upon a plane, the point m is necessarily upon a sphere, (Note at the end of the Memoir;) this point m lies, therefore, on the intersection of the second given plane and this sphere. Hence, the cone generated by the moving right line is cut in circular sections by the two fixed planes; which proves the theorem.

54. A plane and a right line being given, if round their point of intersection we conceive a plane to turn which makes, with the given plane and right line, angles, the ratio of whose sines is constant, this plane will envelope a cone of the second degree, which will have the given plane for one of its cyclic planes; and the given right line will be the polar of this cyclic plane with relation to the cone.

A right line and a plane being given, if round their point of intersection we conceive a right line to turn which makes, with the given right line and plane, angles, the ratio of whose sines is constant, this right line will generate a cone of the second degree, in which the given right line will be a focal line, and the given plane will be the director plane corresponding to that focal line.

These two propositions are the converses of the two theorems (34.)

It is sufficient to prove the first.

For this purpose, let us draw a transversal plane parallel to the given plane; it will meet the given right line in a point A, and the moveable plane in a right line D.

The sine of the angle which this moveable plane makes with the fixed right line is equal to the perpendicular let fall from the point A upon the moveable plane divided by AO (O being the point of intersection of the given plane and right line, round which the moveable plane turns;) the sine of the angle which this moveable plane makes with the transversal plane is equal to the same perpendicular divided by the distance of the point A from the right line D. The ratio of the two sines is therefore equal to this distance of the point A from the right line D, divided by the distance OA; this ratio is constant by hypothesis; the distance OA is likewise constant; consequently, the distance of the point A from the right line D is constant; thus, the trace of the moveable plane upon the transversal plane is always at the same distance from the point A; which proves that this trace envelopes a circle whose centre is at the point A; the cone enveloping the moveable plane is therefore one of the second degree, having a cyclic plane parallel to the transversal plane, and the right line OA is the polar of this cyclic plane; which proves the theorem.

55. In the second of the two preceding theorems it is to be observed, that if from a point in the moveable right line perpendiculars be let fall on the given right line and plane, the ratio of these perpendiculars will be the same as the ratio of the sines of the angles which the moveable right line makes with the given right line and plane. We have, therefore, the following theorem:

The geometric locus of a point whose distances from a fixed right line and plane are in a given ratio, is a cone of the second degree, which has the given right line and plane for a focal line and its corresponding director plane.

If the ratio of the distances be one of equality, we infer that

The surface, in which every point is equidistant from a given right line and plane, is a cone of the second degree, having the given right line and plane for a focal line and its corresponding director plane.

M. Hachette has employed this theorem in the descriptive solution of the question: "to find the centre of a sphere touching a plane and circumscribed about a hyperboloid of revolution." (See M. Quetelet's Correspondance Mathématique et Physique, Vol. IV. p. 285.) As M. Hachette's proof is extremely simple, we subjoin it.

Let a transversal plane be drawn parallel to the given plane, and let a right circular cylinder be described, having for its axis the given right line, and for its radius the distance of the transversal plane from the given plane. The transversal plane intersects the cylinder in an ellipse, every point of which is equidistant from the given right line and plane. Hence we perceive, that on the cone whose base is this ellipse, and whose vertex is the point of concourse of the given right line and plane, every point is equidistant from this right line and plane.

SECTION V.

PROBLEMS RELATING TO THE CYCLIC PLANES AND TO THE FOCAL LINES OF CONES OF THE SECOND DEGREE; AND GENERAL PROPERTIES OF TRIHEDRAL AND TETRAHEDRAL ANGLES.

56. When the vertex and a cyclic plane of a cone of the second degree are given, only three other conditions are required to determine this cone.

For, if a transversal plane be drawn parallel to the given cyclic plane, this plane will intersect the cone in a circle, which it will be necessary to determine. For this purpose we require three conditions. Hence, the modes of describing a circle subject to three given conditions, will become applicable to the analogous questions relating to a cone of the second degree, of which we have the vertex and one cyclic plane given.

By reference to the supplementary cone, it appears that the vertex and a focal line of a cone of the second degree being given, we require three other conditions to determine this cone; and the modes of construction, will correspond to those relating to the analogous questions on cones of the second degree, of which the vertex and a cyclic plane are given.

57. The centre of a circle which passes through two given points is upon the perpendicular erected at the middle point of the right line connecting the two points. Hence, in accordance with what has been already said (39) we deduce the two following theorems:

If a cyclic plane and two sides of a cone of the second degree be given, the polar of this cyclic plane is upon a plane determined by the two following conditions:

1. Its trace upon the plane of the two sides must be the harmonic conjugate, with relation to the two sides, of the right line in which the plane of these two sides intersects the cyclic plane.

2. Its trace upon the cyclic plane must be perpendicular to this right line of intersection.

If a focal line and two tangent planes of a cone of the second degree be given, the director plane corresponding to this focal line, will pass through a right line which is the intersection of the two following planes:

1. The plane which passes through the intersection of the two given tangent planes, and is the harmonic conjugate, with relation to these two planes, of the plane passing through this intersection and through the focal line.

2. The plane passing through this focal line at right angles with this latter plane.

58. The centre of a circle touching two right lines is upon the right line which bisects the angle or the supplement of the angle between them. Hence we infer, that

If a cyclic plane and two tangent planes of a cone of the second degree be given, the polar of this cyclic plane lies in the plane which passes through the intersection of the two tangent planes, and whose trace upon the cyclic plane bisects the angle or the supplement of the angle between the traces of the two tangent planes upon this cyclic plane.

If a focal line and two sides of a cone of the second degree be given, the director plane corresponding to this focal line passes through the right line in which the plane of the two sides meets the vector plane bisecting the angle or the supplement of the angle between the two vector planes passing through the focal line and through the two sides.

59. *Problem.*—Given three sides and a cyclic plane of a cone of the second degree, to determine the polar of this cyclic plane.

Problem.—Given three tangent planes and a focal line of a cone of the second degree, to determine the director plane corresponding to this focal line.

The solutions of these two problems are evidently furnished by the two theorems (57;) it is, therefore, unnecessary to give them.

Problem.—Given three tangent planes and a cyclic plane of a cone of the second degree, to determine the polar of that cyclic plane.

Problem.—Given three sides and a focal line of a cone of the second degree, to determine the director plane of the cone corresponding to that focal line.

The solutions of these two problems are evidently furnished by the two theorems (58;) it is, therefore, unnecessary for us to give them. But it ought to be remarked, that just as the problem of describing a circle touching three given right lines has four solutions, so each of these two problems will likewise admit of four solutions.

61. From what has been said it is plain, that the solution of this question, "to describe a circle touching three circles lying in the same plane," furnishes immediately the solutions of the two following problems:

Problem.—Given three cones of the second degree, having the same vertex and a common cyclic plane, to describe a fourth cone of the second degree, touching these three cones, and having the same cyclic plane with them.

Problem.—Given three cones of the second degree, having the same vertex and the same focal line, to determine a fourth cone of the second degree touching these three cones, and having the same focal line with them.

Each of these problems, generally speaking, admits of eight solutions. The given cones may become planes or right lines, in the same way as we have shewn, (Annales de Mathématiques de M. Gergonne,) that, in the construction of a circle touching three given circles, these latter may become right lines or points.

62. *We have seen* (59,) *that when we have a trihedral angle and a plane passing through its vertex, this plane may be considered as the cyclic plane of a cone of the second degree, three of whose generatrices are the three edges of the trihedral angle; consequently, we infer from theorem* 22, *that*

If, being given a trihedral angle and a transversal plane passing through its vertex, we draw in each face of this angle a right line through the vertex, which makes, with one of the two edges lying in this face, an angle equal to that contained between the other edge and the trace of the transversal plane upon this face, the three right lines thus drawn in the three faces are in the same plane.

This plane and the given transversal plane are the cyclic planes of a cone of the second degree, circumscribed about the trihedral angle.

We have seen (59,) *that when we have a trihedral angle and a right line passing through its vertex, this right line may be considered as the focal line of a cone of the second degree touching the three faces of the trihedral angle; consequently, we infer from theorem* 22, *that*

If, being given a trihedral angle and a right line passing through its vertex, we draw through each edge of this trihedral angle a plane, making, with one of the two faces adjacent to this edge, an angle equal to that which the other face makes with the plane determined by the same edge and by the given right line, the three planes thus drawn pass through the same right line.

This right line and the given right line are the focal lines of a cone of the second degree inscribed in the trihedral angle.

63. *This theorem leads to the solution of the following problem:*

Problem.—Given three sides and cyclic plane of a cone of the second degree, to determine the second cyclic plane of this cone.

This theorem leads to the solution of the following problem:

Problem.—Given three tangent planes and a focal line of a cone of the second degree, to determine the second focal line of this cone.

64. Theorems (28) give rise to the two following:

A right line being drawn through the vertex of a trihedral angle, if it be projected orthogonally upon the three faces of the angle, and through the three projections a cone be made to pass, one of whose cyclic planes is perpendicular to the given right line, this cone will intersect the three faces of the trihedral angle in three new right lines, and the planes passing through these right lines and respectively perpendicular to the three faces will intersect in the same right line, which will be perpendicular to the second cyclic plane of the cone.

A plane being drawn through the vertex of a trihedral angle, if in this plane we draw three right lines passing through the vertex, and respectively perpendicular to the three edges of the angle, each edge and its perpendicular will determine a plane; if a cone of the second degree be described touching the three planes thus determined, and having for a focal line the perpendicular to the given plane, through the edges of the trihedral angle three new tangent planes to the cone may be drawn, and the right lines drawn in these planes respectively perpendicular to the three edges, will all three be in a new plane perpendicular to the second focal line of the cone.

65. Theorem (30) leads to the following:

If from any point perpendiculars be let fall upon the three faces of a trihedral angle, and through each edge of the angle a plane be drawn perpendicular to the right line joining the feet of the perpendiculars upon the two faces adjacent to this edge, the three planes thus drawn will intersect in the same right line.

66. Theorems (31) lead to the following:

A right line being drawn through the vertex of a trihedral angle, its orthogonal projections on the faces of this angle will be the edges of a second trihedral angle; the planes, passing through the edges of the first angle, and respectively perpendicular to the faces of the second, will pass through the same right line.

A plane being drawn through the vertex of a trihedral angle, three right lines drawn in this plane perpendicular to the three edges of the angle will respectively determine with these edges three planes forming a second trihedral angle; the three right lines drawn in the faces of the first angle, and respectively perpendicular to the three edges of the second, will all three be in the same plane.

67. It is known that the circle circumscribed about a triangle, whose three sides touch a parabola, passes through the

focus of that curve. (Traité des Propriétés Projectives de M. Poncelet, p. 268.) Hence we infer the first, and consequently, the second, of the two following theorems:

If about a cone of the second degree several trihedral angles be circumscribed, and as many cones of the second degree be respectively circumscribed about these trihedral angles, having all of them a plane touching the given cone for their common cyclic plane, all these cones will pass through a common generatrix.

If in a cone of the second degree several trihedral angles be inscribed, and as many cones of the second degree be respectively inscribed in these trihedral angles, having all of them a side of the given cone for their common focal line, all these cones will touch the same plane.

68. *From what precedes we infer by means of theorem* 28, *second column, that*

A cone of the second degree being given, if any trihedral angle be circumscribed about it, and in a fixed plane touching the cone three right lines be taken making right angles with the three edges of the trihedral angle, and if we further conceive a cone touching the planes of these three angles and having for its focal line the perpendicular to the fixed tangent plane, this new cone will touch a fixed plane, whatever be the trihedral angle circumscribed about the given cone.

From what precedes we infer by means of theorem 28, *first column, that*

A cone of the second degree being given, if any trihedral angle be inscribed in it, and through a fixed side of the cone three planes be drawn respectively perpendicular to the faces of this angle, and if we further conceive a cone of the second degree to be described passing through their three right lines of intersection with these faces, and having one of its cyclic planes perpendicular to the fixed side of the given cone, this new cone will pass through a fixed right line, whatever be the trihedral angle inscribed in the given cone.

69. These two theorems lead to the two following properties of tetrahedral angles:

If a tetrahedral angle be given, and also, a fixed plane passing through its vertex, the planes of the four faces of the tetrahedral angle, taken three by three, will form four trihedral angles: now, if in the given plane right lines be taken making right angles with the edges of these trihedral angles, and a cone of the second degree be described, touching the planes of the three right angles corresponding to each trihedral angle, and having one of its focal lines perpendicular to the given plane, the four cones thus determined will all touch the same plane.

If a tetrahedral angle be given, and also, a fixed right line passing through its vertex, the four edges of this angle, taken three by three, will determine four trihedral angles: now, if through the given right line planes be drawn perpendicular to the faces of each of these trihedral angles, and through the right lines of intersection of these planes with these faces a cone of the second degree be described, having one of its cyclic planes perpendicular to the given right line, the four cones thus determined will pass through a common generatrix.

SECTION VI.

ORGANIC DESCRIPTION OF CONES OF THE SECOND DEGREE.

70. The organic description of the conic sections given by Newton, is founded upon the following theorem:

If any two constant angles turn round two fixed points as vertices, so that two of their sides intersect upon a given right line, the point of intersection of their two other sides will generate a conic section, which will pass through the two fixed points. (See Universal Arithmetic, Vol. i.; and Principia Mathematica, Lib. i, Lemma 21.)

This construction of the conic sections by points, is general.

The cones of the second degree admit of a similar construction, by which their sides are determined; and also of another analogous construction by which their tangent planes may be determined.

These two modes of describing the cones of the second degree are included in the two following theorems:

71. If any two dihedral angles of invariable magnitude whose edges are fixed and meet one another, turn round these edges so that two of their faces intersect on a fixed plane passing through the point of intersection of the two edges, the intersection of their two other faces will generate a cone of the second degree which will pass through the two fixed edges.

For let us conceive a cone of the second degree, whose focal lines are the edges of the two moveable dihedral angles, and which touches the gived fixed plane.

The two first faces of the two dihedral angles intersect upon this plane, by hypothesis; through their right line of intersection, let us draw a plane M, *touching the cone; this plane meets the second face of the first angle in a right line which generates a plane* P, *touching the cone.* (44, *second column.*)

In like manner, the plane M *meets the second face of the second angle in a right line which generates another plane* P′, *touching the cone.*

If any two plane angles of invariable magnitude have for their common vertex a fixed point round which they turn in two given planes, in such a manner that the plane determined by two of their sides turns round a fixed right line passing through their common vertex, the plane determined by their two other sides will envelope a cone of the second degree, which will touch the two planes in which the two angles respectively move.

For let us conceive a cone of the second degree, whose cyclic planes are the two planes in which the two angles move, and which passes through the fixed right line.

The plane which contains the two first sides of the two angles turns round this right line, and cuts the cone along another side M; *the plane determined by this side and by the second side of the first angle turns round fixed side of the cone* P. (44, *first column.*)

In like manner, the plane determined by the side M *and by the second side of the second angle turns round another fixed side of the cone* P′.

Hence the two second faces of the two dihedral angles meet any plane M *touching the cone in two right lines contained in two fixed planes* P, P′, *touching this cone; but these two faces respectively pass through the two edges of the dihedral angles which are the focal lines of the cone; therefore, their right line of intersection generates a cone of the second degree, passing through these two fixed edges.* (51, *second column.*) Q. E. D.

Hence the two second sides of the two angles are respectively in two planes which turn round two fixed sides of the cone P, P′, *and which intersect upon another indefinite side* M; *but these two sides are respectively in the two cyclic planes of the cone; therefore, their plane will envelope a cone of the second degree, touching these two cyclic planes.* (51, *first column.*) Q. E. D.

It would have been sufficient to demonstrate one of these two theorems, since the other might have been deduced from it by reference to the supplementary cone.

The first part of our proof of the first theorem is analogous to the method which M. Poncelet has employed in proving the theorem of Newton. (Traité des Propriétés Projectives, p. 274.) It is satisfactory to make this comparison, since it furnishes an example of the advantages attending the modes of proof adopted in that learned work, when applied to questions in the geometry of three dimensions as well as to those of plane geometry.

72. *Problem.*—Given five sides of a cone of the second degree, to determine all the other sides of the cone by the movement of two dihedral angles round their edges.

Let A, B, C, D, E, *be the five given sides; let us take the first two* A, B, *as the edges of the two moveable dihedral angles, and conceive that when two faces of these angles simultaneously coincide with the plane of the two sides* A, B, *their two other faces intersect along the third side* C; *so that these two angles are perfectly determined.*

Now let them be turned round their edges A, B, *so that their two faces, which before passed through the side* C, *may intersect along the fourth side* D, *and again along the fifth side* E; *their two first faces which originally coincided with the plane of the two edges* A, B, *will successively intersect in two right lines* D′, E′.

Let the right line of intersection

Problem.—Given five tangent planes to a cone of the second degree, to determine all the other tangent planes by the movement of two angles in their respective planes.

Let A, B, C, D, E, *be the five given planes; let us take the first two* A, B, *as the planes of the two moveable angles; the third plane* C, *will intersect them in two right lines which will make two angles with the right line of intersection with these two planes; these will be the two moveable angles.*

Now let them be turned round their common vertex in their respective planes A, B, *so that their two sides, which before were contained in the plane* C, *may lie in the plane* D, *and again in the plane* E; *their two first sides which originally coincided with the right line of intersection of the two planes* A, B, *will successively determine two planes* D′, E′.

Let the plane of the same two

of the same two faces traverse the plane determined by the two right lines D′, E′, *the right line of intersection of the two other faces will generate the required cone ; as appears from the preceding theorem.*

sides turn round the right line of intersection of these two planes D′, E′, *the plane of the two other sides of the two moveable angles will assume all the positions of the planes touching the required cone ; as appears from the preceding theorem.*

SECTION VII.

PROPERTIES OF HYPERBOLOIDS DEDUCED FROM THOSE OF CONES OF THE SECOND DEGREE.

73. It is known that a hyperboloid, whether of one or of two sheets, and its asymptotic cone have the same systems of conjugate diameters. Now, we have proved that in every cone of the second degree there are two axes such that two conjugate planes passing through either of them, are always at right angles (14) ; it follows, therefore, that

In every hyperboloid, whether of one or of two sheets, there are two diameters such that two conjugate planes passing through either of them are always at right angles (*a*).

These two diameters are the focal lines of the asymptotic cone of the hyperboloid.

74. These right lines lie within the cone ; consequently they meet the hyperboloid if it be one of two sheets, and do not meet it if it be of one sheet. About a hyperboloid of one sheet, two cylinders might therefore be circumscribed, having their generatrices parallel to the two right lines in question.

Two planes passing through either of these right lines, and mutually conjugate with relation to the hyperboloid, will also be conjugate with relation to the circumscribed cylinder, whose generatrices are parallel to this right line. A transversal plane perpendicular to this right line will cut the cylinder in a conic section, and will intersect the two conjugate planes along two conjugate diameters of that curve ; now these two diameters will be at right angles ; the conic section will therefore be a circle ; whence we infer, that

About every hyperboloid of one sheet two cylinders may be circumscribed, whose bases upon planes perpendicular to their generatrices are circles (*b*).

(*a*) There are two similar right lines in the ellipsoid ; as we shall show hereafter, when proving a more general property of the surfaces of the second degree.

(*b*) This equally applies to the ellipsoid.

The axes of these cylinders are the focal lines of the asymptotic cone of the hyperboloid.

75. These cylinders possess two characteristic properties which are easily deduced from two properties of cones of the second degree. For it is well known that the sides of the asymptotic cone of a hyperboloid of one sheet are parallel to the generatrices of the hyperboloid; we infer, therefore, from theorem (34,) second column, in consequence of what has been just stated, that,

In every hyperboloid of one sheet, the ratio of the sines of the angles made by each generatrix with the axis of either of the two circumscribed right circular cylinders, and with the diametral plane conjugate to that axis, is constant.

76. Theorem (24,) second column, leads to the following:

In every hyperboloid of one sheet, the sum or the difference of the angles made by each generatrix with the two axes of the circumscribed right circular cylinders, is constant.

77. The properties of the cones of the second degree lead also to two theorems relative to the subcontrary sections of the hyperboloid of one sheet.

For every transversal plane intersects a hyperboloid, and its asymptotic cone in two similar, and similarly placed conic sections. This may be easily shown, for where two surfaces of the second degree touch one another along a plane curve, every plane intersects them in two conic sections, which have a double contact on the right line in which this plane meets the plane of the curve of contact of the two surfaces. If the plane of this curve be at an infinite distance, the two conic sections will have a double contact on a right line lying at an infinite distance, which indicates that the two conic sections are similar, similarly placed, and concentric.

Hence, it follows, that the planes of the circular sections of a hyperboloid are parallel to the cyclic planes of its asymptotic cone.

78. Consequently, theorem (26,) first column, leads to the following:

In every hyperboloid of one sheet, the product of the sines of the angles made by each generatrix with the planes of the subcontrary sections, is constant.

79. Every plane touching a hyperboloid of one sheet passes through two generatrices, and the plane passing through the centre of the hyperboloid intersects the asymptotic cone along two sides parallel to these two generatrices; consequently, theorem (22,) first column, leads to the following:

In every hyperboloid of one sheet, the tangents drawn at any point on its surface to the two circular sections passing *through this point respectively make equal angles with the two generatrices which meet in this point.*

Note referred to in Nos. 17 *and* 53.

Theorem.—*If from a fixed point radii be drawn, terminating at different points of a given plane, and points be assumed on these radii, whose distances from the fixed point are reciprocally proportional to the radii, these points will all be on a sphere which will pass through the fixed point, and whose centre will be on the perpendicular let fall from this point upon the plane.*

For, let O be the fixed point, P the foot of the perpendicular let fall from this point upon the given plane, and M any point whatever in the plane; let us assume upon the radii OP, OM, two points p, m, such that op, om, may be reciprocally proportional to OP, OM, that is to say, such that we shall have $OP.\,op = a^2$, $OM.\,om = a^2$; (a being a constant right line;) the triangles OPM opm, are plainly similar, the angle m is, therefore, a right angle; which proves that the point m is upon the sphere described upon op as diameter. Q. E. D.

MEMOIR

ON

THE GENERAL PROPERTIES

OF THE

SPHERICAL CONICS.

ON

THE GENERAL PROPERTIES

OF

THE SPHERICAL CONICS.

SECTION I.

PRELIMINARY CONSIDERATIONS.

1. Let thére be a cone of the second degree having its vertex at the centre of a sphere. The curve of intersection of these two surfaces is a *line of curvature* of the cone, since it is every where normal to the generatrices of the cone, which are also lines of curvature.

By the intersection of the sphere with the two sheets of the cone, we obtain two curves, which return into themselves, and which are evidently symmetrical with relation to the three principal planes of the cone, in the same manner as the cone itself is symmetrical with relation to these three planes. These two curves are also symmetrical with relation to the three great circles in which these planes intersect the sphere.

Each of these two curves is of double curvature, unless the cone be one of revolution, in which case they are both circles of the sphere. These two curves, in the general case of any cone of the second degree, have the form of an ellipse, and differ less from this curve as the radius of the sphere becomes greater. On this account we may be permitted to call each of these curves a *spherical ellipse*.

2. Let the principal plane of the cone be drawn, and let us only consider the hemisphere and the sheet of the cone situated above this plane; this sheet will determine a spherical ellipse upon the surface of the sphere; the principal axis of the cone will penetrate the sphere in a point which will be the *centre* of this ellipse; for it will bisect every arc of a great circle passing through this point and included within the curve; since the arc of a great circle, included between

two points on a sphere, measures the angle contained by two right lines drawn from the vertex of the cone to these two points.

Every arc of a great circle passing through the centre and terminated by the curve is a *diametral arc* of the curve.

The greatest and the least of the diametral arcs of the ellipse, which we shall call its *principal diametral arcs*, are contained in the planes of the greatest and least sections of the cone. These arcs terminate in four points of the ellipse, which may be called its *vertices*.

The two focal lines of the cone will penetrate the hemisphere, on which is traced the spherical ellipse which we are now considering, in two points which may be called the *foci* of the ellipse, on account of the similarity which may be shown to exist between their properties and those of the foci of plane ellipses.

Lastly, the two cyclic planes of the cone will intersect the same hemisphere in two great semi-circles, which will have the major axis of the cone for their common diameter. This major axis lies in the plane of the greatest diametral arc of the ellipse; and these two great semi-circles are perpendicular to the least diametral arc, and never meet the ellipse. In order to indicate their origin, let us call these two arcs the *cyclic arcs* of the spherical ellipse.

3. Now let us draw the plane of the least section of the cone. It is clear that this plane will divide the complete intersection of the cone and sphere into two equal parts, symmetrically placed with relation to this plane.

Let us consider the part of this intersection lying on one side of this plane: it will consist of two branches which will be halves of the two spherical ellipses; these two branches being symmetrical with respect to the diametral plane perpendicular to the principal axis of the cone, and receding more and more from this plane as the distance from their vertices increases, form, when considered together, a curve which may be called a *spherical hyperbola*. Its *centre* is the point where the major axis of the cone penetrates the hemisphere on which the curve is traced; its two *foci* are the points where the two focal lines of the cone penetrate this hemisphere. The curve has but two vertices, and its foci lie upon the arc of a great circle which joins them.

Lastly, the two cyclic planes of the cone intersect the hemisphere in two great semicircles, which pass through the centre of the hyperbola, and make equal angles with the arc which joins its two foci. These are the two *cyclic arcs* of the hyperbola.

4. Again, let us consider the hemisphere lying on either side of the plane of the greatest section of the cone, that is, the plane containing the two focal lines.

On this hemisphere we shall have two halves of the two spherical ellipses. These two curves turn their concave parts towards each other, and approach towards the principal plane of the cone as the distance from their vertices increases.

These two branches taken together form a third species of spherical curve. This curve has a *centre*, which is the point of intersection of the sphere with the minor axis of the cone; it has four *foci* which are in the plane of the greatest section of the cone, and two *cyclic arcs* which lie between the two branches of the curve, and are perpendicular to the arc of the great circle which joins its two vertices.

The three curves which we have just been considering, are portions of the same curve which arises from the complete intersection of the sphere with a cone of the second degree, having its vertex at the centre of the sphere. We may, therefore, designate them all by the common name of *spherical conics*.

5. *The Spherical conics* possess a great number of properties, of which the most part are very remarkable.

It is to be observed, that *all these properties are double*, that is to say, to every proposition relative to the spherical conics, there always corresponds a second proposition relative to these same curves.

This results from the fact, that the properties of the cones of the second degree are double, as we have already proved. (See section 10 of the preceding Memoir.) But we may also prove this general principle by observing, that to any figure traced upon the surface of the sphere there always corresponds a second figure, which is the envelope of the arcs of great circles, whose planes are perpendicular to the radii of the sphere drawn to the different points of the first figure. Each property of the first figure has, therefore, always a corresponding property in the second figure; but if the first figure be a *conic*, the second will also be a *conic*, arising from the intersection of the sphere with the cone *supplementary* to that on which the first conic lies; which proves the principle that has been stated.

We may call the two conics *supplementary*, as well as the two cones; the cyclic arcs of either of them are in the diametral planes perpendicular to the diameters of the sphere passing through the foci of the other.

6. Among the numerous properties of the spherical conics

there are two which have been already given by M. Magnus, of Berlin. (See Annales de Mathématiques, August, 1825.) They are the two following:

"The sum or the difference of the two radii vectores drawn from the two foci of a spherical conic to any point in it is constant.

"The two radii vectores drawn to any point of the conic make equal angles with the arc of a great circle touching the curve at that point."

7. It is evident that, by reference to the supplementary conic whose properties we have just stated, we might immediately deduce from these two propositions the following theorems which appear to us to possess an equal degree of interest:

"In every spherical conic there are two arcs of great circles such that the sum or the difference of the angles, which each arc of a great circle touching the conic makes with them, is constant.

"Every arc of a great circle, touching the conic and terminated by these two fixed arcs, is bisected at its point of contact with the conic."

The former of these two propositions shows that,

"The envelope of the bases of all the spherical triangles, which have a common vertical angle and the same area, is a spherical conic."

8. This latter, and the preceding theorem, are exactly analogous to the following well-known properties of the hyperbola:

"The envelope of the bases of all the plane triangles, which have a common vertical angle and the same area, is a hyperbola.

"The portion of any tangent to a hyperbola intercepted by the two asymptotes is bisected at the point of contact with the curve."

We shall find the same analogy between several other properties of the spherical conics and those of the hyperbola, so that the cyclic arcs of the spherical conics will be found to bear the same relation to those curves that its asymptotes do to the hyperbola.

As to the two theorems of M. Magnus, they bear a striking similarity to the known properties of the foci of the conic sections: the same may be observed of all the other properties of the foci of these curves.

In fact, we shall show in the last section of this Memoir, how the properties of the foci of the conic sections and those of the asymptotes of the hyperbola may be considered as consequences from those which we prove with relation to the foci and the cyclic arcs of the spherical conics.

But as these properties of the foci of the spherical conics may be applied to all the plane conic sections, it might naturally be supposed that those of the cyclic arcs ought, in like manner, to have corresponding ones in every plane conic section; they therefore guide us to the discovery of a new class of general properties of the plane conic sections, of which those of the asymptotes of the hyperbola are only particular cases.

We shall make these new properties of the plane conic sections, the subject of a separate memoir, as we mean to couple with them the analogous properties of the surfaces of the second degree.

9. It is surprising that the elegance of M. Magnus's two propositions has not yet directed the attention of geometers to researches of this kind, and that the theory of the spherical conics has yet to be constructed, although, in consequence of the duality of all the propositions of spherical geometry, it admits perhaps of a wider extension than that of the plane conic sections.

We do not, indeed, pretend to lay before the reader a theory of these conics; but we propose merely to state a certain number of those among their properties which relate to the *foci* and to the *cyclic arcs*, and which result immediately from those which we have proved relative to the cones of the second degree. It is manifest that various other known properties of the cones of the second degree, would furnish, in like manner, and without difficulty, properties of the spherical conics, which ought to find a place in a treatise on these curves.

10. A spherical ellipse and a spherical hyperbola have each only two foci; but a complete spherical conic has four foci placed at the extremities of two diameters of the sphere. In the theorems relating to two foci, we must always be understood to refer to two foci assumed respectively on these two diameters, and not to two foci assumed upon the same diameter; for the two foci ought to belong to the two focal lines of the cone on which the conic is traced.

But it will be more simple to suppose, in all that follows, that the conic is a single spherical ellipse or hyperbola, and to consider upon the sphere only the hemisphere on which this ellipse or hyperbola is traced; by this means we shall avoid all ambiguity; an arc of a great circle touching the curve will have only one point of contact with it, whilst it would touch it in two points, were we to consider the complete conic; any two arcs of great circles will intersect in only one point, since we consider only one-half of the sphere; for the same reason three arcs of great circles will intersect two by two in only three points, and will form but a single spherical triangle.

As we shall only have to speak of arcs of great circles, we may be allowed, for the sake of brevity, to use merely the word *arc;* and it must be fully understood that we only mean arcs of great circles.

We shall call the angle formed by two arcs of great circles a *spherical angle;* their point of intersection will be the *vertex* of the angle. We shall call every arc of a great circle passing through the focus of a conic a *vector arc.*

The properties of the spherical conics which we are about to state being all immediate consequences from those of the cones of the second degree, which we have proved in the preceding Memoir, it will be sufficient to give the enunciations of them, indicating, in the case of each, the corresponding property of the cones of the second degree (*a*).

SECTION II.

PROPERTIES RELATING TO THE TWO CYCLIC ARCS OF A SPHERICAL CONIC; AND PROPERTIES RELATING TO ITS TWO FOCI.

11. The two theorems (20 *a*,) lead to the following:

Every arc of a great circle touching a spherical conic intersects the two cyclic arcs in two points, which are equally distant from the point of contact with this tangent arc.

The vector arcs, drawn from the two foci of a spherical conic to any point on the curve, make equal angles with the arc of a great circle touching the conic at that point.

12. Conversely,

If a curve traced upon a sphere be such that every arc of a great circle, touching the curve and terminated by two fixed arcs of great circles, is bisected at the point where it touches the curve, this curve is a spherical conic.

If a curve traced upon a sphere be such that the arcs of great circles, drawn from two fixed points to any point in the curve, make equal angles with the arc of a great circle touching the curve at that point, this curve is a spherical conic.

This follows from the two theorems (21 *a*.)

13. The two theorems (22 *a*,) lead to the following, of which the two in No. 11 are only particular cases:

Every arc of a great circle intersects a spherical conic in two points

The two vector arcs, drawn from the two foci of a spherical conic to

(*a*) We shall use numbers, followed by the letter *a*, in referring to the paragraphs of the preceding Memoir.

which are equally distant from the points in which this arc respectively cuts the two cyclic arcs.

the point of intersection of two arcs touching the curve, respectively make equal angles with these tangent arcs.

14. Theorems (23 *a*,) lead to the following :

The planes of two arcs touching a spherical conic intersect the planes of the two cyclic arcs in four right lines, which are the generatrices of a right cone whose axis of revolution is perpendicular to the plane of the great circle passing through the two points of contact of the tangent arcs.

The planes of four vector arcs drawn from the two foci of a spherical conic to any two points of the curve will all touch the same right cone whose axis of revolution is the right line of intersection of the planes of the two arcs touching the conic at these two points.

15. Theorems (24 *a*,) lead to the following :

The sum or the difference of the angles which each arc touching a spherical conic makes with the two cyclic arcs is constant.

The sum or the difference of the vector arcs drawn from the two foci of a spherical conic to any point of the curve is constant.

16. Theorems (25 *a*,) lead to the following :

Every arc of a great circle touching a spherical conic intersects the two cyclic arcs in two points, such that the product of the trigonometric tangents of the semi-arcs lying between these points and the point of intersection of the two cyclic arcs is constant.

The vector arcs, drawn from the two foci of a spherical conic to any point of the curve, make, with the diametral arc which joins the two foci, two angles, such that the product of the trigonometric tangents of their halves is constant.

17. Theorems (26 *a*,) lead to the following :

In every spherical conic the product of the sines of the arcs of great circles drawn from any point of the curve at right angles with the two cyclic arcs is constant.

In every spherical conic the product of the sines of the arcs of great circles drawn from the two foci at right angles with any arc touching the curve is constant.

18. The two theorems (28 *a*, second and first columns,) respectively lead to the two following :

A spherical conic and its two cyclic arcs being given, if one extremity of an arc of 90° traverse either of the two cyclic arcs whilst the other extremity moves along the conic, this quadrantal arc will envelope a second spherical conic,

If from the foci of a spherical conic arcs be drawn perpendicular to the arcs touching the curve, their respective points of intersection with these tangent arcs will be upon a second spherical conic, which will have a double contact with the given which will have a double contact with the given one, and whose foci will be the extremities of the radii of the sphere perpendicular to the planes of the cyclic arcs of the given conic.

one, and whose cyclic arcs will be in the planes perpendicular to the radii of the sphere which pass through the two foci of the given conic.

19. The two theorems (31 *a*, second and first columns,) respectively lead to the following:

A spherical conic and one of its cyclic arcs being given, if between this arc and the curve, two arcs of 90° be inserted, and from their point of intersection a third quadrantal arc be drawn terminating in the arc which joins those two extremities of the two former ones which lie upon the conic, this extremity of the third arc will fall upon the second cyclic arc of the conic.

If from a focus of a spherical conic two arcs be drawn perpendicular to two arcs touching the curve, and if the feet of the two perpendicular arcs be joined by an arc of a great circle, the arc drawn at right angles with this latter from the point of concourse of the two tangent arcs will pass through the second focus of the curve.

20. Theorems (32 *a*,) lead to the following:

If two spherical conics have the same cyclic arcs, and a common tangent arc be drawn to them, the part of this arc intercepted between the two points of contact will be a quadrant.

If two spherical conics which have the same foci intersect, they will be at right angles to each other at each point of intersection.

SECTION III.

PROPERTIES OF THE SPHERICAL CONICS RELATING TO A SINGLE CYCLIC ARC; AND PROPERTIES RELATING TO A SINGLE FOCUS.

21. If through any point on the surface of a sphere two arcs of great circles be drawn touching a spherical conic, the arc of a great circle joining the two points of contact, may be called the *polar arc* of the point with relation to the conic; and conversely, this point will be called the *pole* of its polar arc.

From the properties of right lines and polar planes in cones of the second degree which have been stated (1 *a*,) it evidently follows, that

The polar arcs of all the points of any arc of a great circle, with relation to a spherical conic, pass all of them through the same point, which is the pole of that arc; and conversely, *The poles of all the arcs of great circles passing through the same point, taken with relation to a spherical conic, are all of them upon the same arc of a great circle which is the polar arc of the fixed point.*

To the polar arcs of its foci we give the name of *director arcs* of a spherical conic.

22. Hence, the two theorems (34, *a*,) lead to the following:

In every spherical conic, the sine of the angle which each tangent arc to the curve makes with a cyclic arc has a constant ratio to the sine of the distance of this tangent arc from the pole of the cyclic arc.

In every spherical conic, the ratio of the sines of the arcs which measure the distances of each point on the curve from a focus and from its corresponding director arc is constant.

23. Theorems (36, *a*,) lead to the following:

Every arc touching a spherical conic, and the arc drawn through its point of contact and through the pole of a cyclic arc of the conic, meet that cyclic arc in two points the distance between which is 90°.

The two vector arcs drawn from one focus of a spherical conic to any point of the curve and to the point where its tangent arc at the former point meets the director arc, are always at right angles.

24. Theorems (37, *a*,) lead to the following:

Two tangent arcs to a spherical conic, and the arc which joins their points of contact with the curve, intersect the cyclic arc in three points, the third of which bisects the distance between the first two.

The vector arcs drawn from a focus of a spherical conic to two points of the curve make equal angles with the vector arc drawn to the point of intersection of the two arcs touching the conic at those two points.

25. Theorems (38, *a*,) lead to the following:

Two tangent arcs to a spherical conic, and the arc passing through their point of intersection and through the pole of a cyclic arc, meet that arc in three points the third of which bisects the distance between the other two.

The vector arcs drawn from a focus of a spherical conic to two points on the curve make equal angles with the vector arc drawn to the point in which the arc joining the two points on the curve meets the director arc.

26. Theorems (39, *a*,) lead to the following:

The arc passing through the pole of a cyclic arc of a spherical conic, and through the point of intersection of two arcs touching the curve, and the arc passing through the two points of contact of these tangent arcs, meet the cyclic arc in two points, the distance between which is 90°.

The two vector arcs drawn from one focus of a spherical conic to the point of intersection of two arcs touching the curve, and to the point in which the arc passing through the two points of contact meets the director arc, are at right angles.

The points in which the arc joining the two points of contact meets the cyclic arc, and the arc which joins its pole with the point of concourse of the two tangent arcs, are harmonic conjugates with relation to the two points of contact.

The two arcs passing through the point of concourse of the two tangent arcs, and passing, the one through the focus of the conic, and the other through the point in which the arc which joins the two points of contact meets the director arc, are harmonic conjugates with relation to the two tangent arcs.

27. The two theorems (40, *a*,) lead to the following:

If through a point assumed upon a cyclic arc of a spherical conic two arcs be drawn touching the conic, the arc joining the two points of contact will meet the cyclic arc in a point whose distance from the former is 90°.

If through a focus of a spherical conic a transversal arc be drawn, its pole with relation to the conic will be upon the director arc, and the arc drawn from the focus to this pole will be perpendicular to the transversal arc.

28. Theorems (41, *a*,) lead to the following:

When a spherical quadrilateral is inscribed in a spherical conic, the portion of a cyclic arc of the conic included between two adjacent sides of the quadrilateral is supplemental to the arc included between the two other sides.

When a spherical quadrilateral is circumscribed about a spherical conic, the angle between the two vector arcs drawn from one focus to two adjacent vertices of the quadrilateral is supplemental to the angle between the two vector arcs drawn to the two other vertices.

29. Theorems (42, *a*,) lead to the following:

If through two fixed points on a spherical conic two arcs be drawn which intersect in any third point of the curve, the segment which they will intercept upon a cyclic arc will be of invariable magnitude.

This segment will be a quadrant, if the arc which joins the two fixed points passes through the pole of the cyclic arc.

Two fixed arcs being drawn touching a spherical conic, and any third tangent arc intersecting the two former in two points, the vector arcs drawn from a focus of the conic to these two points will contain between them a constant angle.

This angle will be right, if the point of concourse of the two fixed tangent arcs be upon the director arc corresponding to the focus.

30. The two theorems (43, *a*,) lead to the following:

If through the two vertices which are at the extremities of the least diametral arc of a spherical ellipse two arcs be drawn intersecting in any third point of the curve, the segment intercepted between these two arcs upon a cyclic arc will be a quadrant.

Every arc touching a spherical conic cuts the arcs touching the curve at the two vertices which are at the extremities of its greatest diametral arc in two points such that the two vector arcs drawn from a focus to these two points are at right angles.

31. Theorems (44, *a*,) lead to the following:

If upon a cyclic arc of a spherical conic a segment of given magnitude be arbitrarily assumed, and through one of its extremities and a fixed point of the curve an arc be drawn meeting the curve in a second point, the arc drawn through this second point and through the other extremity of the segment will pass through a fixed point on the conic.

If round a focus of a spherical conic, as vertex, a spherical angle of invariable magnitude be made to turn, and through the point where one of its sides meets a fixed arc touching the curve, a second tangent arc be drawn, this arc will meet the second side of the angle in a point the geometric locus of which will be an arc touching the conic.

32. Theorems (45, *a*,) lead to the following:

If through a point assumed arbitrarily on a cyclic arc of a spherical conic two arcs be drawn touching the curve, the sum of the trigonometric co-tangents of the angles which they make with the cyclic arc will be constant.

If through a focus of a spherical conic an arc be drawn arbitrarily meeting it in two points, the sum of the trigonometric co-tangents of the arcs lying between the foci and these two points will be constant.

SECTION IV.

GEOMETRIC LOCI RELATING TO THE CYCLIC ARCS AND TO THE FOCI OF THE SPHERICAL CONICS.

33. Theorems (46 *a*,) lead to the following:

If the sides of a spherical angle of variable magnitude pass always through two fixed points on the surface of a sphere, whilst the segment intercepted between its sides upon an arc of a given great circle is of a constant length, the vertex of this angle will generate a spherical conic which will have the fixed arc for a cyclic arc, and which will pass through the two fixed points.

Two fixed arcs and a point being given on a sphere, if round this point as vertex a spherical angle of invariable magnitude be made to turn, and if the two points in which its sides respectively meet the two fixed arcs be joined by an arc of a great circle, this arc will envelope a spherical conic which will have the fixed point for a focus, and which will touch the two fixed arcs.

34. Theorems (47 *a*,) lead to the following:

If two tangent arcs be drawn to a spherical conic so that the segment intercepted between them upon a cyclic arc of the conic may be of a

If an angle of invariable magnitude be made to turn round a focus of a spherical conic as vertex, the arc joining the two points in which its

constant length, the geometric locus of the point of concourse of these two arcs will be a second spherical conic.

The arc joining the two points of contact of these two arcs with the given conic will envelope a third conic.

The cyclic arc in question will be a cyclic arc of the two new conics, and this arc will have the same pole with relation to the three conics.

sides meet the curve will envelope a second conic.

The tangent arcs to the given conic at these two points will intersect upon a third conic.

The focus in question will also be a focus of the two new conics, and the corresponding director arc will be the same in the three conics.

35. Theorems (48, *a*,) lead to the following:

If round a fixed point assumed on a spherical conic a spherical angle of variable magnitude be made to turn, whose sides intercept upon a cyclic arc of the curve a segment of a constant length, the arc joining the two points in which the sides of this angle meet the conic will envelope a second conic; the cyclic arc upon which the segments are measured will be a cyclic arc of the new conic, and it will have the same pole in the two curves.

If round a focus of a spherical conic as vertex a spherical angle of invariable magnitude be to turn, and through the two points in which its sides meet a fixed arc touching the conic two arcs be drawn touching the curve, the point of concourse of these two arcs will generate a second conic; the focus of the given conic will also be a focus of the new conic, and the corresponding director arc will be the same in the two curves.

36. Theorems (49, *a*,) lead to the following:

A spherical conic and a fixed arc arbitrarily drawn being given, if two tangent arcs to the conic be drawn so that the segment intercepted between them on the given arc may be a quadrant, the point of concourse of these two arcs will generate a second conic which will have the given arc for a cyclic arc.

This arc will have the same pole in the two conics.

37. *If the fixed arc be one of the principal diametral arcs of the conic, the theorem may be thus stated:*

If a variable spherical angle, circumscribed about a spherical conic, move so that the segment intercepted between its sides upon a principal diametral arc of the conic may always be a quadrant, the vertex of this angle will generate a small circle of the sphere.

A spherical conic and a fixed point arbitrarily assumed on the sphere being given, if round this point as vertex a right spherical angle be made to turn, and if the points in which its sides meet the conic, taken two by two, be joined by four arcs, these four arcs will envelope a second conic of which the fixed point will be a focus.

This point will have the same polar arc in the two curves.

If the fixed point be the centre of the conic, the theorem may be thus stated:

If a right spherical angle turn round the centre of a spherical conic as vertex, the arc joining the points in which its two sides meet the curve will envelope a small circle of the sphere.

This is also a consequence from the two theorems (50, *a.*)

38. The two theorems (51, *a*,) lead to the following:

If round two fixed points on a spherical conic two arcs be made to turn intersecting in any third point of the curve, these arcs will respectively meet the two cyclic arcs of the conic in two points, and the arc joining these two points will envelope a spherical conic touching these two cyclic arcs.

Two fixed tangent arcs being drawn to a spherical conic, any third tangent arc will intersect them in two points, and the arcs respectively drawn through these points and through the two foci of the conic will intersect in a point the geometric locus of which will be a spherical conic passing through the two foci of the given one.

39. Theorems (52, *a*,) lead to the following:

If round two fixed points two arcs be made to turn, containing between them a right angle, their point of intersection will generate a spherical conic passing through the two fixed points, and whose cyclic arcs will be in the two planes perpendicular to the radii of the sphere drawn to the two fixed points.

If the extremities of an arc of 90° move along the sides of any given spherical angle, this moveable arc will envelope a spherical conic which will touch the two sides of the angle, and whose foci will be the extremities of the radii of the sphere perpendicular to the planes of these sides.

40. Theorems (53, *a*,) lead to the following:

Two fixed arcs being given upon a sphere, if a point be sought such that the product of the sines of its distances from the two fixed arcs may be constant, the geometric locus of this point will be a spherical conic whose cyclic arcs will be the two given arcs.

Two fixed points being given upon a sphere, if an arc be drawn such that the product of the sines of its distances from these two points may be constant, this arc will envelope a spherical conic whose foci will be the two given points.

41. Theorems (54, *a*,) lead to the following:

An arc and a point being given upon a sphere, if an arc be sought such that the sine of the angle which it makes with the given arc, and the sine of its distance from the given point, may have a constant ratio, this arc will envelope a conic which will have the given arc for a cyclic arc; and the given point will be, with relation to this conic, the pole of that cyclic arc.

A point and an arc being given upon a sphere, if a point be sought such that the sines of its distances from the given point and arc may have a constant ratio, the geometric locus of this point will be a conic of which the given point will be a focus; and the given arc will be the director arc corresponding to that focus.

42. If in the second theorem the ratio of the sines be one of equality, we infer from it, that

The spherical curve every point of which is equi-distant from a given point and from a given arc of a great circle is a conic having this point and arc for a focus and its corresponding director arc.

SECTION V.

PROBLEMS RELATING TO THE CYCLIC ARCS AND TO THE FOCI OF THE SPHERICAL CONICS; AND GENERAL PROPERTIES OF SPHERICAL TRIANGLES AND QUADRILATERALS.

43. When a cyclic arc of a spherical conic is given, only three other conditions are required in order to determine the curve.

When a focus of a spherical conic is given, only three other conditions are required in order to determine the curve.

This is a consequence of what we have said with reference to cones of the second degree (56, *a*.)

44. Theorems (57, *a*,) lead to the following:

A cyclic arc and two points of a spherical conic being given, the pole of this cyclic arc lies upon the arc which passes through the two following points:

1. The point which is, with relation to the two given points, the harmonic conjugate of that in which the arc joining these two points meets the given cyclic arc.

2. The point on the cyclic arc which is 90° distant from that in which this arc meets the arc joining the two given points.

A focus and two tangent arcs to a spherical conic being given, the director arc corresponding to that focus passes through the point of intersection of the two following arcs:

1. The arc which passes through the point of concourse of the two given tangents, and which is the harmonic conjugate, with relation to these two arcs, of the arc drawn through this point of concourse and through the given focus.

2. The arc drawn through this focus perpendicular to this latter arc joining the focus with the point of concourse of the two tangent arcs.

45. Theorems (58, *a*,) lead to the following:

A cyclic arc and two tangent arcs to a spherical conic being given,

The pole of that cyclic arc lies upon the arc drawn through the point of concourse of the two tangent arcs and through the middle of the arc (or the supplement of the arc) intercepted upon this cyclic

A focus and two points on a spherical conic being given,

The director arc corresponding to that focus passes through the point in which the arc joining the two given points meets the vector arc which bisects the angle (or the supplement of the angle) contained arc between the two given tangent arcs.

between the two vector arcs drawn from the given focus to the two points on the conic.

46. The two theorems (44,) respectively contain the solutions of the two following problems :

Problem.—Given three points and a cyclic arc of a spherical conic, to determine the pole of this cyclic arc.

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the director arc corresponding to this focus.

47. The two theorems (45,) in like manner enable us to resolve the two following problems, each of which admits of four solutions:

Problem.—Given three tangent arcs and a cyclic arc of a spherical conic, to determine the pole of this cyclic arc.

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the director arc corresponding to that focus.

48. We have just seen (46,) that

A spherical triangle being given, and also any arc of a great circle, this arc may be considered as a cyclic arc of a spherical conic passing through the three vertices of the triangle.

A spherical triangle being given, and also a fixed point upon the sphere, this point may be considered as the focus of a spherical conic touching the three sides of the triangle.

This remark will aid us in the proof of some general properties of spherical triangles and quadrilaterals.

49. Theorems (13,) lead, as appears from what we have just said, to the two following properties of spherical triangles:

If a spherical triangle and a transversal arc be given, and upon each side of the triangle a point be assumed whose distance from one extremity of that side is equal to the distance of the other extremity from the point in which the transversal arc meets the side, the three points thus determined upon the three sides of the triangle will lie upon the same arc of a great circle.

A spherical triangle being given, if through a fixed point an arc be drawn to each vertex of the triangle, and through that vertex a second arc be drawn which makes with one of the adjacent sides of the triangle an angle equal to that which the first arc makes with the other side, the three arcs thus drawn will pass through the same point.

This arc and the given transversal arc will be the cyclic arcs of a conic circumscribed about the spherical triangle.

50. *This theorem furnishes the solution of the following*

Problem.—Given three points and a cyclic arc of a spherical conic, to determine the second cyclic arc of this curve.

This point and the given point will be the foci of a spherical conic inscribed in the given triangle.

This theorem furnishes the solution of the following

Problem.—Given three tangent arcs and a focus of a spherical conic, to determine the second focus of this curve.

51. Theorems (18) lead to the two following properties of spherical triangles:

A spherical triangle and a transversal arc being given, if upon this arc three points be assumed which are respectively distant by 90° from the three vertices of the triangle, and these points be connected with the three vertices by three arcs, if we now describe a conic touching these last three arcs and having for its focus the extremity of the radius of the sphere perpendicular to the plane of the given transversal arc, and through the vertices of the given triangle draw three new arcs touching this conic, and if we assume upon these arcs three points respectively distant by 90° from the vertices, these three points will be upon the same arc of a great circle.

This arc will be in the plane perpendicular to the radius of the sphere drawn to the second focus of the conic.

If from a point assumed arbitrarily upon a sphere three arcs be drawn perpendicular to the three sides of a spherical triangle, and through the feet of these perpendiculars a spherical conic be made to pass, which has for a cyclic arc the great circle contained in the plane perpendicular to the radius of the sphere drawn to the given point, this conic will meet the three sides of the triangle in three new points such that the arcs drawn through these points and respectively perpendicular to the three sides will pass through the same point.

This point will be the extremity of the radius perpendicular to the plane of the second cyclic arc of the conic.

52. Theorems (19) lead to the two following properties of spherical triangles:

A triangle and an arc of a great circle being traced upon a sphere, if through each vertex of the triangle an arc of 90° be drawn terminated by the given arc, the three arcs thus drawn will form a second spherical triangle; if through the vertices of this new triangle three arcs of 90° be drawn, respectively terminated by the three opposite sides of the first triangle, the extremities of these three arcs will lie upon the same arc of a great circle.

If from a point assumed upon a sphere arcs be drawn perpendicular to the three sides of a spherical triangle, their feet will be the three vertices of a new triangle inscribed in the first; and if from the vertices of the first triangle arcs be drawn respectively perpendicular to the opposite sides of the second triangle, these three arcs will pass through the same point.

53. The two theorems (67 *a*,) lead to the following:

If about a spherical conic any number of spherical triangles be circumscribed, and as many spherical conics be circumscribed about these triangles, having all of them for a common cyclic arc a fixed tangent arc to the given conic, all these curves will pass through the same point.

If in a spherical conic any number of spherical triangles be inscribed, and as many spherical conics be inscribed in these triangles, having all of them for a common focus, a fixed point on the given conic, all these curves will touch the same arc of a great circle.

54. From the two theorems (68 *a*,) we deduce the following:

A spherical conic being given, if any spherical triangle be circumscribed about it, and if upon a fixed arc touching the conic three points be assumed such that the arcs respectively drawn from these points to the three vertices of the triangle may be quadrants; and if we further suppose a conic to be described touching these three arcs, and having for its focus the extremity of the radius of the sphere perpendicular to the plane of the fixed arc, this new conic will always touch the same arc of a great circle, whatever be the triangle circumscribed about the given conic.

If from a fixed point assumed upon a spherical conic, three arcs be drawn perpendicular to the sides of a spherical triangle inscribed in the conic, and if through the feet of these perpendiculars a spherical conic be made to pass, having for a cyclic arc the arc of a great circle lying in the plane perpendicular to the radius of the sphere drawn to the fixed point on the given conic, this new conic will always pass through a fixed point, whatever be the triangle inscribed in the given conic.

55. The two theorems (69 *a*,) lead to the two following general properties of spherical quadrilaterals:

A spherical quadrilateral being given, its sides, taken three by three, will form four triangles; if through the vertices of each of these triangles three quadrantal arcs be drawn, terminated by the same given arc of a great circle, and if a conic be described, touching these three arcs, and having for a focus the extremity of the radius of the sphere perpendicular to the plane of the given great circle, the four conics thus determined will all touch the same arc of a great circle.

A spherical quadrilateral being given, its four vertices, taken three by three, will determine four triangles; if from a fixed point arcs be drawn perpendicular to the three sides of each of these triangles, and if through the three points in which these arcs respectively meet the sides, a spherical conic be made to pass, having one of its cyclic arcs in the plane perpendicular to the radius of the sphere which is drawn to the given fixed point, the four conics thus determined will all pass through the same point.

SECTION VI.

ORGANIC DESCRIPTION OF THE SPHERICAL CONICS.

56. The two theorems (71 *a*,) relating to the description of the cones of the second degree, respectively lead to the two following:

If any two spherical angles, each of invariable magnitude, turn round two fixed points as vertices, so that two of their sides intersect on a given fixed arc, the point of intersection of their two other sides will generate a spherical conic which will pass through the two fixed vertices of the moveable angles.

If along two given fixed arcs any two segments, each of invariable magnitude, be made to move, so that the arc joining two of their extremities may turn round a fixed point, the arc joining their two other extremities will envelope a conic which will touch the two fixed arcs along which the two segments move.

The first of these two theorems is exactly analogous to that of Newton, relative to the organic description of the plane conic sections.

57. *Problem.*—Given five points of a spherical conic, to determine all the other points of the curve by the movement of two spherical angles round their vertices.

Let A, B, C, D, E, *be the five given points; let us take the first two* A, B, *for the vertices of the two moveable angles, and these angles will be* (CAB), (CBA).

Now let these angles be turned round their vertices A, B, *so that their sides* CA, CB, *may pass at the same time through the point* D, *and again through the point* E; *their two other sides, which originally coincided with the arc* AB, *will successively intersect in two points* D′, E′.

Problem.—Given five tangent arcs to a spherical conic, to determine all the other tangent arcs of the curve by the movement of two arcs along the circumferences of two great circles.

Let A, B, C, D, E, *be the five given arcs; let us take the circumferences of the great circles to which the first two,* A, B, *belong, for those along which the two moveable arcs are to be measured; the third arc* C, *produced if necessary, will meet the first two* A, B, *in two points, and the arcs included between these two points and the point of intersection of* A *and* B *will be the two moveable arcs.*

Now let these two arcs be moved along their circumferences, so that their extremities, which before were placed upon the arc C, *may be upon the arc* D, *and again upon the arc* E; *their two other extremities, which originally coincided with the point of intersection of the two arcs* A, B, *will successively determine two arcs* D′, E′.

Let the point of intersection of the same two sides traverse the arc of a great circle determined by the two points D′, E′; *the point of intersection of the two other sides will generate the required conic, as appears from the preceding theorem.*

Let the arc joining the same two extremities turn round the point of intersection of these two arcs D′, E′; *the arc joining the two other extremities of the two moveable arcs will envelope the required conic, as appears from the preceding theorem.*

SECTION VII.

PROPERTIES OF THE PLANE CONIC SECTIONS DEDUCED AS CONSEQUENCES FROM THOSE OF THE SPHERICAL CONICS.

58. We have stated (8), that from the propositions relative to the spherical conics, contained in this memoir, we might deduce a very great number of the properties of the foci of the plane conic sections, and some properties of the asymptotes of the hyperbola.

For this purpose, it is sufficient to suppose that the centre of the sphere recedes to an infinite distance upon the radius which passes through the centre of the spherical conic. This curve will degenerate into a plane conic which will be an ellipse or a hyperbola; and the properties of the foci of the spherical conics will become those of the plane conics.

In the case where the conic becomes a hyperbola, the cyclic arcs will become two fixed right lines drawn through the centre of the curve; and the properties of the cyclic arcs will apply to these two right lines, which each of these properties leads us to recognize as the asymptotes of the hyperbola.

59. We shall state the various theorems which may be deduced in this way from the properties of the spherical conics; at the end of each we shall refer to the number of the theorem from which it is a consequence. First, we shall give those which relate to the foci, and afterwards those which relate to the asymptotes.

These two classes of theorems which are thus found to have a common origin, present a remarkable connexion between the properties of the foci and those of the asymptotes; properties so different both in their statement and in the proofs usually given of them. When discussing the new properties of the conic sections of which we have spoken (8), we mean to shew how close a relation subsists in other respects between the foci of the conic sections and the asymptotes of the hyperbola.

I.

GENERAL PROPERTIES OF THE TWO FOCI OF THE PLANE CONIC SECTIONS CONSIDERED SIMULTANEOUSLY.

1. The radii vectores drawn from the two foci to any point of the conic section make equal angles with the tangent at that point (11).

2. Conversely: If a curve be such that the radii vectores drawn from two fixed points to each point on it make equal angles with the tangent at that point, the curve is a conic section (12).

3. The two radii vectores drawn from the two foci of a conic section to the point of concourse of two tangents respectively make equal angles with those tangents (13).

4. The four radii vectores drawn from the two foci of a conic section to any two points on it are tangents to the same circle, whose centre is the point of concourse of the tangents to the conic section at those two points (14).

5. The sum or the difference of the two radii vectores drawn from the two foci of a conic section to any point on it is constant (15).

6. The product of the trigonometric tangents of the semi-angles, which the two radii vectores, drawn from the two foci of the conic section to any point on it, make with its major axis, is constant (16).

7. The rectangle under the perpendiculars let fall from the two foci of a conic section on each tangent to the curve is constant (17).

8. The locus of the feet of the perpendiculars, let fall from the two foci of a conic section upon its tangents, is a circle (18).

9. If from a focus of a conic section perpendiculars be let fall upon two tangents to the curve, and if a right line be drawn joining their feet, the perpendicular drawn to this right line from the point of concourse of the two tangents will pass through the second focus of the curve (19).

10. If two conic sections which have the same foci cut one another, they are at right angles to each other at each point of intersection (20).

II.

PROPERTIES OF THE PLANE CONIC SECTIONS RELATING TO A SINGLE FOCUS.

1. The ratio of the distances of each point on a conic section from a focus and from the corresponding directrix is constant (22).

2. The radii vectores drawn from a focus of a conic section to a point on the curve, and to the point where the tangent at that former point meets the directrix, are at right angles (23).

3. The radii vectores drawn from a focus of a conic section to two points on the curve make equal angles with the radius vector drawn to the point of concourse of the two tangents at those points (24).

4. The two radii vectores, drawn from a focus of a conic section to two points on the curve, make equal angles with the radius vector drawn to the point in which the chord joining the two points on the curve meets the directrix (25).

5. The radius vector drawn from a focus of a conic section to the point of concourse of two tangents to the curve, and the radius vector drawn to the point in which the chord joining the two points of contact meets the directrix, are at right angles.

The two right lines drawn through the point of concourse of the two tangents, and passing, the one through the focus, and the other through the point in which the chord joining the two points of contact meets the directrix, are harmonic conjugates with relation to the two tangents (26).

6. If a transversal be drawn through the focus of a conic section, its pole, with relation to the curve, will be upon the directrix, and the radius vector drawn from the focus to this pole will be perpendicular to the transversal (27).

7. When a quadrilateral is circumscribed about a conic section, two opposite sides of the quadrilateral subtend at either focus two angles supplemental one to the other (28).

8. Two fixed tangents being drawn to a conic section, the part of another moveable tangent intercepted between the two former ones will subtend at either focus an angle of invariable magnitude.

This angle will be right, if the point of concourse of the two fixed tangents be upon the directrix (29).

9. The portion of any tangent of a conic section intercepted between the two tangents drawn at the extremities of the major axis subtends a right angle at either focus (30).

10. A conic section and a fixed tangent to it being given, if round either focus, as vertex, an angle of invariable magnitude be made to turn, and through the point in which one of its sides meets the fixed tangent we draw a second tangent to the curve, this second tangent will meet the second side of the angle in a point, the geometric locus of which will be a tangent to the conic section (31).

11. Any chord of a conic section passing through a focus

is divided at that point into two parts the sum of the reciprocals of which is constant (32).

III.

GEOMETRIC LOCI RELATING TO THE FOCI OF THE PLANE CONIC SECTIONS.

1. If an angle of invariable magnitude be made to turn round a fixed point as vertex, and if a right line be drawn joining the two points in which its two sides respectively meet two given right lines, this right line will envelope a conic section whose focus will be the vertex of the moveable angle, and which will touch the two given right lines (33).

2. If an angle of invariable magnitude be made to turn round the focus of a conic section as vertex, the chord which it will subtend will envelope a second conic section.

The tangents to the given conic section at the extremities of this chord will intersect in a point the geometric locus of which will be a third conic section.

These two new conic sections will have the same focus as the given conic section, and the same corresponding directrix (34).

3. If an angle of given magnitude be made to turn round the focus of a conic section as vertex, and through the points in which its sides meet a tangent to the curve, two new tangents be drawn, their point of concourse will generate a second conic section, of which the fixed vertex of the moveable angle will be a focus, and the corresponding directrix will be that of the given conic section (35).

4. If in the plane of a conic section a right angle be made to turn round a fixed point as vertex, the chord subtended in the conic section by the sides of this angle will envelope a second conic section, one of whose foci will be at the fixed point, and the corresponding directrix will be the polar of this point with relation to the given conic section (36).

If the vertex of the moveable angle be the centre of the given conic section, the second conic section will be a circle(37).

5. A conic section and two fixed tangents being given, if any third tangent be drawn, and through the points in which it meets the two fixed tangents, two right lines be drawn respectively passing through the two foci of the curve, these two right lines will intersect in a point, the geometric locus of which will be a conic section passing through the two foci of the given one (38).

6. Two fixed points being given, if a right line be drawn, such that the rectangle under its distances from the two given fixed points shall be constant, this right line, and all the others determined in like manner, will envelope a conic section whose foci will be the two fixed points (40).

7. A point and a right line being given, the geometric locus of a point whose distances from the given point and right line are to each other in a constant ratio, is a conic section in which the given point is a focus, and the given right line is the corresponding directrix (41).

IV.

PROBLEMS RELATING TO THE FOCI OF THE PLANE CONIC SECTIONS, AND GENERAL PROPERTIES OF PLANE TRIANGLES AND QUDRILATERALS.

1. When one of the foci of a conic section is given, only three other conditions are required to determine this curve. (43).

2. A focus and two tangents of a conic section being given, the directrix corresponding to that focus passes through the point of intersection of the two following right lines:

a. The right line which passes through the point of concourse of the two given tangents, and is the harmonic conjugate, with relation to these two tangents, of the right line drawn from this point of concourse to the focus of the curve:

b. The right line drawn through the focus perpendicular to the right line joining this focus with the point of concourse of the two given tangents (44).

3. A focus and two points on a conic section being given, the directrix corresponding to that focus passes through the point in which the right line joining the two given points meets the right line which bisects the angle, or the supplement of the angle, contained between the two radii vectores drawn from the focus to the two given points (45).

4. The last theorem but one furnishes the solution of the following problem:

Given three tangents and one of the foci of a conic section, to determine the directrix corresponding to that focus.

5. The last theorem enables us to resolve the following problem, which admits of four solutions:

Given three points and one of the foci of a conic section, to determine the directrix corresponding to that focus.

6. Three right lines being drawn from a fixed point to the three vertices of a triangle, if through each vertex a new right

line be drawn, making with one of the two sides adjacent to that vertex an angle equal to that which the former right line makes with the other side, the three right lines thus drawn will pass through the same point.

This point and that from which the three former right lines were drawn, will be the foci of a conic section inscribed in the given triangle (49).

7. This theorem furnishes a solution of the following problem:

Given three tangents and one of the foci of a conic section, to determine the other focus.

8. If from a point assumed arbitrarily in the plane of a triangle three perpendiculars be let fall upon its sides, and a circle be described passing through their feet, the perpendiculars drawn to the sides of the triangle at the three new points in which this circle meets them, will all pass through the same point (51).

9. If from a point assumed in the plane of a triangle three perpendiculars be let fall upon its sides, their feet will be the vertices of a second triangle inscribed in the former; and if through the vertices of the first triangle right lines be drawn respectively perpendicular to the opposite sides of the second, these three right lines will pass through the same point (52).

The feet of the perpendiculars let fall from this new point upon the sides of the given triangle, and those of the three former perpendiculars, will be six points lying on the circumference of the same circle.

10. If any number of triangles be inscribed in a conic section, and as many conic sections be again inscribed in these triangles, having all of them for their common focus a fixed point on the given conic section, all these curves will touch the same right line (53).

11. If from a point assumed upon a conic section perpendiculars be let fall upon the sides of a triangle inscribed in the conic section, the circle described through the feet of these perpendiculars will pass through a fixed point, whatever be the triangle inscribed in the conic section (54).

12. The four vertices of a quadrilateral, taken three by three, determine four triangles; if from any point perpendiculars be let fall upon the sides of each triangle, and a circle be described passing through their feet, the four circles thus determined will pass through the same point (55).

V.

PROPERTIES OF THE TWO ASYMPTOTES OF THE HYPERBOLA CONSIDERED SIMULTANEOUSLY.

1. Every tangent to a hyperbola meets the asymptotes in two points, which are equally distant from the point of contact (11).

2. Conversely: If a curve be such that the portion of each tangent intercepted between two given right lines is bisected at the point of contact, this curve is a hyperbola whose asymptotes are the two given right lines (12).

3. The portions of any secant intercepted between the hyperbola and its asymptotes are equal (13).

4. Every tangent to a hyperbola meets the asymptotes in two points, the rectangle under the distances of which from the centre of the curve is constant (16).

5. In every hyperbola the rectangle under the distances of any point on the curve from the two asymptotes is constant (17).

VI.

PROPERTIES OF THE HYPERBOLA RELATING TO A SINGLE ASYMPTOTE.

1. Two tangents to a hyperbola, and the right line joining the points of contact, meet an asymptote in three points, the third of which bisects the distance between the first two (24).

2. If the two sides of a variable angle, whose vertex traverses a hyperbola, pass through two fixed points on the curve, the segment intercepted between the sides of this angle upon an asymptote will be of a constant length (29).

3. If on either asymptote of a hyperbola a portion of given length be arbitrarily assumed, and through one of its extremities and a fixed point on the curve a right line be drawn meeting the curve in a second point, the right line joining this second point with the second extremity of the portion assumed on the asymptote will turn round a fixed point on the hyperbola (31).

VII.

GEOMETRIC LOCI RELATING TO THE ASYMPTOTES OF THE HYPERBOLA.

1. If the two sides of an angle of variable magnitude pass always through two fixed points, and intercept upon a given right line a segment of a constant length, the vertex of this angle will generate a hyperbola, which will pass through the two fixed points, and which will have the given right line for an asymptote (33).

2. If two tangents to a hyperbola intercept between them on one of the asymptotes a segment of a constant length, the geometric locus of their point of concourse will be a second hyperbola.

The chord joining the two points of contact will envelope a third hyperbola.

The asymptote on which are measured the segments intercepted between the tangents will be an asymptote of the two new hyperbolas (34).

3. If an angle of variable magnitude be made to turn round a fixed point on a hyperbola as vertex, intercepting on an asymptote a segment of a constant length, the chord subtended by this angle will envelope a second hyperbola, to which the right line on which the intercepted segments are measured will also be an asymptote (35).

4. If the sides of a variable angle pass always through two fixed points on a hyperbola, whilst its vertex traverses the curve, the sides of this angle will respectively meet the two asymptotes in two points, and the right line joining these two points will envelope a conic section touching the two asymptotes of the hyperbola (38).

5. Two right lines being given, the geometric locus of the point, the rectangle under whose distances from the two right lines is constant, will be a hyperbola of which the two right lines are the asymptotes (40).

VIII.

GEOMETRIC LOCI RELATING TO ANY CONIC SECTIONS.

The first three of the five preceding theorems give rise to new theorems, by means of the method of transforming geometrical relations which we have explained in a preceding Memoir; these theorems, which relate to any conic section, are the following:

"1. If an angle of variable magnitude be made to turn round a fixed point as vertex, so that the segment which it intercepts upon a fixed axis may be of a constant length, the right line joining the two points in which the sides of this angle respectively meet two given right lines will envelope a conic section, which will touch these two right lines, and which will pass through the vertex of the moveable angle.

"The tangent to the curve at this point will be parallel to the axis on which the segments are measured."

This proposition furnishes the solution of the following problem:

"Given four tangents to a conic section, and the point of contact of one of these tangents, to determine all the other tangents of the curve, by the continued movement of a segment of a constant length along a fixed right line."

"2. If an angle of variable magnitude be made to turn round a point on a conic section as vertex, so that the segment which it intercepts upon a fixed axis parallel to the tangent at that point shall be of a constant length, the chord subtended by this angle in the conic section will envelope a second conic section.

"The point of concourse of the two tangents to the first conic section at the extremities of this chord will generate a third conic section.

"These two new curves will touch the given one at the vertex of the moveable angle."

"3. If an angle of variable magnitude be made to turn round a fixed point on a conic section as vertex, whilst its sides intercept a segment of a constant length upon a parallel to the right line touching the curve at that point, and if through the points in which the sides of this angle meet a fixed tangent to the curve two new tangents be drawn, their point of concourse will generate a conic section, which will touch the given one at the vertex of the moveable angle."

IX.

PROBLEM RELATING TO THE ASYMPTOTES OF THE HYPERBOLA, AND GENERAL PROPERTIES OF TRIANGLES.

1. One of the asymptotes of a hyperbola being given, only three other conditions are required to determine the curve (43).

2. If any transversal be drawn in the plane of a triangle, and on each side a point be assumed, whose distance from one of the two extremities of that side is equal to the distance of the other extremity from the point in which the transversal

VII.

GEOMETRIC LOCI RELATING TO THE ASYMPTOTES OF THE HYPERBOLA.

1. If the two sides of an angle of variable magnitude pass always through two fixed points, and intercept upon a given right line a segment of a constant length, the vertex of this angle will generate a hyperbola, which will pass through the two fixed points, and which will have the given right line for an asymptote (33).

2. If two tangents to a hyperbola intercept between them on one of the asymptotes a segment of a constant length, the geometric locus of their point of concourse will be a second hyperbola.

The chord joining the two points of contact will envelope a third hyperbola.

The asymptote on which are measured the segments intercepted between the tangents will be an asymptote of the two new hyperbolas (34).

3. If an angle of variable magnitude be made to turn round a fixed point on a hyperbola as vertex, intercepting on an asymptote a segment of a constant length, the chord subtended by this angle will envelope a second hyperbola, to which the right line on which the intercepted segments are measured will also be an asymptote (35).

4. If the sides of a variable angle pass always through two fixed points on a hyperbola, whilst its vertex traverses the curve, the sides of this angle will respectively meet the two asymptotes in two points, and the right line joining these two points will envelope a conic section touching the two asymptotes of the hyperbola (38).

5. Two right lines being given, the geometric locus of the point, the rectangle under whose distances from the two right lines is constant, will be a hyperbola of which the two right lines are the asymptotes (40).

VIII.

GEOMETRIC LOCI RELATING TO ANY CONIC SECTIONS.

The first three of the five preceding theorems give rise to new theorems, by means of the method of transforming geometrical relations which we have explained in a preceding Memoir; these theorems, which relate to any conic section, are the following :

"1. If an angle of variable magnitude be made to turn round a fixed point as vertex, so that the segment which it intercepts upon a fixed axis may be of a constant length, the right line joining the two points in which the sides of this angle respectively meet two given right lines will envelope a conic section, which will touch these two right lines, and which will pass through the vertex of the moveable angle.

"The tangent to the curve at this point will be parallel to the axis on which the segments are measured."

This proposition furnishes the solution of the following problem:

"Given four tangents to a conic section, and the point of contact of one of these tangents, to determine all the other tangents of the curve, by the continued movement of a segment of a constant length along a fixed right line."

"2. If an angle of variable magnitude be made to turn round a point on a conic section as vertex, so that the segment which it intercepts upon a fixed axis parallel to the tangent at that point shall be of a constant length, the chord subtended by this angle in the conic section will envelope a second conic section.

"The point of concourse of the two tangents to the first conic section at the extremities of this chord will generate a third conic section.

"These two new curves will touch the given one at the vertex of the moveable angle."

"3. If an angle of variable magnitude be made to turn round a fixed point on a conic section as vertex, whilst its sides intercept a segment of a constant length upon a parallel to the right line touching the curve at that point, and if through the points in which the sides of this angle meet a fixed tangent to the curve two new tangents be drawn, their point of concourse will generate a conic section, which will touch the given one at the vertex of the moveable angle."

IX.

PROBLEM RELATING TO THE ASYMPTOTES OF THE HYPERBOLA, AND GENERAL PROPERTIES OF TRIANGLES.

1. One of the asymptotes of a hyperbola being given, only three other conditions are required to determine the curve (43).

2. If any transversal be drawn in the plane of a triangle, and on each side a point be assumed, whose distance from one of the two extremities of that side is equal to the distance of the other extremity from the point in which the transversal

meets that side, the three points thus assumed will be in the same right line (49).

3. This theorem furnishes the solution of the following problem:

Given three points and one of the asymptotes of a hyperbola, to find the second asymptote.

4. If any number of triangles be circumscribed about a hyperbola, and if we conceive as many hyperbolas to be circumscribed about these triangles, having all of them a tangent of the given hyperbola for their common asymptote, all these curves will pass through the same point (53).

5. By means of the method of transforming geometrical relations which we have already employed (No. VIII. in this section) the preceding theorem (2) gives rise to the following general property of triangles:

"If through each vertex of a triangle two right lines be drawn, of which the first passes through a given fixed point, and the second is such that the angles, which these two right lines respectively make with the two sides of the triangle adjacent to the vertex, intercept equal segments upon a fixed transversal, the three right lines thus determined will pass through the same point."

X.

ORGANIC DESCRIPTION OF THE PLANE CONIC SECTIONS.

The two theorems (56) relative to the description of the spherical conics lead to the two following:

1. If two angles of given magnitudes turn round two fixed points as vertices, so that the point of concourse of two of their sides traverses a right line, the point of concourse of their two other sides will generate a conic section, which will pass through the two fixed points.

2. If along two fixed right lines two segments of given lengths be made to move, so that two of their extremities are always in the same right line with a fixed point, the right line joining their two other extremities will envelope a conic section, which will touch the two fixed right lines.

The first of these two theorems is that of Newton, and enables us to describe the conic section by points; the second, which is new, furnishes a very simple construction of the tangents of the conic sections, as we shall presently see in the solution of the following problem:

3. *Problem.* Given five tangents of a conic section, to determine all its other tangents by the movement of two rectilineal segments along two fixed right lines.

Let A, B, C, D, E, be the five given right lines; let us take the first two A, B, as the two fixed right lines along which the two segments are to move, and for the segments themselves let us take the distances of the point of concourse of these two right lines from the points in which they are intersected by the third right line C. These two segments must be made to move respectively along the two right lines A, B, so that their extremities which before lay upon the right line C, may fall upon the right line D, and again upon the right line E; their two other extremities, which at first coincided with the point of intersection of the two right lines A, B, will successively determine two right lines D′, E′.

Let the two segments move so that these same two extremities may always be in the same right line with the point of intersection of the two right lines D′, E′; the right line joining the two other extremities of the two segments will assume all the positions of the tangents to the required conic section.

4. It is unnecessary to explain the construction by which, as a consequence from Newton's theorem, we determine all the points of a conic section subject to the condition of passing through five given points; it would be a mere repetition of what we have already said in the solution of the same question with respect to the spherical conics (57).

5. Before closing this Memoir, we may observe that the theorem (2), which enables us to construct the tangents of the conic sections, gives rise, by means of our method of transforming geometrical relations, to another theorem available in the construction of conic sections by points, and which, for this purpose, might take the place of Newton's theorem.

This new theorem may be thus stated:

"If two angles intercepting segments of constant lengths upon a fixed axis turn round two fixed points as vertices, so that two of their sides always intersect upon a given right line, the point of concourse of their two other sides will generate a conic section passing through the two fixed points."

This theorem may be employed in the same way that Maclaurin, in his Organic Geometry, made use of Newton's theorem.

6. In order to complete this chapter on the organic construction of conic sections we ought to add, that Newton's theorem also gives rise to a theorem relating to the construction of the tangents of the conic section by the movement of two angles of constant magnitudes. This theorem may be obtained by a polar transformation, a circle being used as the auxiliary conic, as M. Poncelet has pointed out in his Memoire Sur la Théorie des Polaires réciproques; we may state it thus:

"If round a fixed point, as vertex, two angles of constant magnitudes be made to turn, so that the points in which two of their sides respectively meet two given right lines are always in the same right line with a given point, the right line joining the points in which the two other sides respectively meet the same two right lines will envelope a conic section which will touch those two right lines."

NOTES AND ADDITIONS.

PAGE 5, § 6.—*If through the vertex of a cone of the second degree right lines be drawn perpendicular to its tangent planes, they will form another cone, which will be of the second degree.*

The analytical proof of this proposition, though not quite so short, may appear to some readers more satisfactory than the geometrical one.

Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

be the equation of a cone; the axes of coordinates being rectangular. Then the equation of a tangent plane at the point, (x', y', z') will be

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}-\frac{z'z}{c^2}=0.$$

Now, the perpendicular to this tangent plane, passing through the origin, makes angles with the axes of x, y, and z, whose cosines are respectively proportional to $\frac{x'}{a^2}$, $\frac{y'}{b^2}$, and $-\frac{z'}{c^2}$; but these cosines are also proportional to the coordinates X, Y, Z, of any point on that perpendicular. Hence we find $a^2X^2+b^2Y^2-c^2Z^2=0$, for the equation of the second or *supplementary* cone, which is of the second degree; and its tangent plane at the point (X, Y, Z) is evidently perpendicular to the side of the given cone passing through the point $(x', y', z'.)$

PAGE 9, § 17.—The theorem stated in this paragraph is a case of the following general one:

If two surfaces of the second order intersect along a plane curve, they will also intersect along a second plane curve, either real or imaginary.

In order to prove this, let us take the plane of the first curve of intersection as the plane of xy: then the equations of the two surfaces being,

$$\text{A}x^2+\text{A}'y^2+\text{A}''z^2+\text{B}yz+\text{B}'xz+\text{B}''xy+\text{C}x+\text{C}'y+\text{C}''z+1=0,$$

$$ax^2+a'y^2+a''z^2+byz+b'xz+b''xy+cx+c'y+c''z+1=0,$$

we must have the coefficients of the terms, independent of z, respectively equal in both; since the traces of the two surfaces upon the plane of xy are identical; that is, we must have,

$$\text{A} = a,\ \text{A}' = a',\ \text{B}'' = b'',\ \text{C} = c,\ \text{C}' = c'.$$

Hence, subtracting and dividing by z, we get,

$$(\text{A}'' - a'')z + (\text{B} - b)y + (\text{B}' - b')x + \text{C}'' - c'' = 0,$$

which is the equation of a second plane, on which the surfaces intersect.

The normal, at any point on the surface of a cone of the second degree, meets the plane of its least section at the centre of the sphere, passing through the two subcontrary circular sections which intersect in the point at which the normal is drawn.

It is easy to see that the centre of any sphere passing through two subcontrary circular sections, must lie in the plane of the least section of the cone: for the planes of the circular sections are perpendicular to that plane, and their centres lie in it. But further, if the two circular sections meet at a point on the surface of the cone, the sphere passing through them must necessarily touch the cone, so that its centre must lie on the normal at that point. Thus, we have proved that the centre of the sphere is at the point where the normal meets the plane of the least section of the cone.

The preceding proposition leads to an easy and direct proof of the first theorem in No. 24, p. 13.

Let VA, VB, be the two sides of the cone contained in the plane of its least section; and let AB, A′B′, be the traces on this plane of the two subcontrary circular sections which pass through a point P on the surface of the cone, if perpendiculars be drawn to AB, A′B′, at their middle points M, M′, they will meet at O, the centre of the sphere passing through the two subcontrary sections.

Now, the plane touching the cone at P will also touch the sphere, and consequently it will be perpendicular to the radius OP: therefore, the angles, which the tangent plane makes with the two planes of circular section, are respectively equal to the angles which OP makes with OM, OM′; or, what is the same thing, to the angles AOM, A′OM′. But either the sum or the difference of these two angles is constant, and equal to the supplement of the angle AVB. Hence we have proved that

The sum or the difference of the angles, which each tangent plane to a cone of the second degree makes with the two cyclic planes, is constant.

It is easy to extend to the surfaces of the second order in general, the construction just given for determining the centre of the sphere passing through two subcontrary circular sections that meet at a point on the surface.

Let

$$\text{M}z^2 + \text{M}'y^2 + \text{M}''x^2 + \text{N}z + \text{N}'y + \text{N}''x = 0 \qquad (1)$$

be the equation of the surfaces of the second order; the axes of coordinates being rectangular and parallel to conjugate diameters, and the origin on the surface.

In this equation, without at all diminishing its generality, we may assume that M is intermediate between M′ and M″, regard being had to the *signs* as well as to the actual magnitude of these coefficients: for this relation between M, M′, M″, may be always obtained by interchanging the names of the coordinates. Let

$$(z-z')^2+(y-y')^2+(x-x')^2-r^2=0 \qquad (2)$$

be the equation of a sphere, in which x', y', z', r, are indeterminate. Multiplying this latter equation by M, and subtracting from the former, we get,

$$\left.\begin{array}{l}(\mathrm{M}'-\mathrm{M})y^2+(\mathrm{M}''-\mathrm{M})x^2+(\mathrm{N}+2\mathrm{M}z')z \\ +(\mathrm{N}'+2\mathrm{M}y')y+(\mathrm{N}''+2\mathrm{M}x')x-\mathrm{M}(z'^2+y'^2+x'^2-r^2)\end{array}\right\}=0. \quad (3)$$

If the indeterminate quantities in this equation, x', y', z', r, be so assumed that it shall represent a system of two planes, these planes must intersect the given surface along two circles. Such a determination of x', y', z', r, may be effected in an infinite number of different ways; but in all cases we must have

$$\mathrm{N}+2\mathrm{M}z'=0; \qquad (4)$$

as appears from comparing equation (3) with the product of the equations of two planes: and this shows that all the planes of the circular sections are perpendicular to the plane of xy.

The coefficient of z in equation (3) being thus made $=0$, the conditions to be fulfilled, in order that it shall represent two planes, are, 1st, that

$$\frac{(\mathrm{N}'+2\mathrm{M}y')^2}{\mathrm{M}'-\mathrm{M}}+\frac{(\mathrm{N}''+2\mathrm{M}x')^2}{\mathrm{M}''-\mathrm{M}}-4\mathrm{M}(z'^2+y'^2+x'^2-r^2)=0; \quad (5)$$

and 2nd, that the product $(\mathrm{M}-\mathrm{M}')$ $(\mathrm{M}-\mathrm{M}'')$ may not be positive. This latter condition is already complied with, since M is intermediate between M′ and M″.

The value of z being determined by equation (4) we still have three quantities, x', y', r, to dispose of, and may at once satisfy equation (5) by putting

$$\mathrm{N}'+2\mathrm{M}y'=0,\ \mathrm{N}''+2\mathrm{M}x'=0,\ z'^2+y'^2+x'^2-r^2=0.$$

By these assumptions, along with equation (4), the equation (3) is reduced to the form,

$$(\mathrm{M}'-\mathrm{M})y^2+(\mathrm{M}''-\mathrm{M})x^2=0,$$

representing two planes perpendicular to the plane of xy, and whose traces on that plane make angles with the axis of x, the tangents of which are $+\sqrt{\frac{M''-M}{M-M'}}$, and $-\sqrt{\frac{M''-M}{M-M'}}$. These values of the tangents must plainly be real.

The values for the coordinates of the centre of the sphere, and for its radius, are next to be considered. The three equations,

$$N + 2Mz' = 0,\quad N' + 2My' = 0,\quad N'' + 2Mx' = 0,$$

show that the centre is on the normal to the given surface at the origin: for x', y', z' are proportional to N'', N', N; and the equation of the tangent plane at the origin is,

$$Nz + N'y + N''x = 0.$$

Moreover, the equation, $N + 2Mz' = 0$, shows that the centre of the sphere is in the principal plane of the surface, parallel to the plane of xy.

Hence, we have proved that in general,

The circular sections, passing through a given point on a surface of the second order, may be determined by the following construction: At the given point let a normal to the surface be drawn to meet the principal plane, to which the circular sections are perpendicular: the point of intersection will be the centre, and the normal itself will be the radius, of the sphere which cuts the given surface in the two circular sections passing through the given point.

In the analysis of this question there is one case which requires particular notice, viz. that of the hyperbolic paraboloid, where M' and M'' have different signs, and $M = 0$. Here we have real values for the tangents of the angles between the planes of circular section and the axis of x: but the values of x', y', z', become infinite, shewing that the sphere degenerates into a plane, and the circular into rectilinear sections.

Confined within the narrow limits of a note, I must either leave to the student, or defer until some other occasion, the fuller discussion of the analytical method which I have here used in determining the circular sections of the surfaces of the second degree. It will be found to conduct easily to many interesting results relative to these sections.

If we are only allowed to assume that the circular sections of any surface of the second degree are perpendicular to one of its principal planes, we may apply to these surfaces in general the geometrical proof given above in the case of the cone. For the sphere which passes through two circular sections that meet at a point on the surface, must touch the surface, and its centre must therefore be on the normal to the surface at that point. And since the two circular sections are perpendicular to a principal plane, the centre of the sphere passing through them must lie in that plane.

PAGE 12, § 22. These two theorems lead to the following properties of cones having the same cyclic planes or the same focal lines.

When two cones of the second degree have a common vertex and the same cyclic planes, if a plane be drawn through their vertex cutting the two cones, the sides along which it meets one cone will respectively make equal angles with the sides of the other cone, through which the secant plane passes.

When two cones of the second degree have a common vertex and the same focal lines, the angle contained between two planes touching the two cones will be equal to that contained between the two other tangent planes which pass through the right line in which the two former tangent planes intersect.

Cones which have the same focal lines are called by M. Chasles *biconfocal cones:* in like manner, cones which have the same cyclic planes may be called *biconcyclic cones.* An investigation of the properties of these latter would be found to guide the student to many new theorems analogous to those relative to similar and similarly placed conic sections.

In future, for the sake of brevity, I shall dispense with the enunciation of those properties of cones of the second degree which admit of being stated as properties of spherical conics: the latter mode of statement being in general more concise, as well as more readily understood.

PAGE 14, § 25.—I have not succeeded in finding the elementary geometrical proofs, which M. Chasles has promised to give, of the two theorems contained in this paragraph. The following demonstration of the theorem in the first column appears to be as simple as could be desired.

Let APB, A′PB′ be two circular sections of the cone, passing through a point P on its surface; let AB, A′B′, their traces on the plane of the least section of the cone, intersect in O; and at P let tangents be drawn to the two circular sections, respectively meeting AB and A′B′ in T and T′. Then OP is parallel to the line in which the cyclic planes intersect, and the tangents PT, PT′, are parallel to the lines in which the tangent plane at P intersects the cyclic planes. But the angles OPT, OPT′, are respectively double the angles PAO, PA′O, whose tangents are $\frac{PO}{AO}$ and $\frac{PO}{A'O}$: and since $PO^2 = AO.OB$, the product of the tangents of the halves of OPT, OPT′, is equal to $\frac{OB}{OA'}$, which is constant: OB and OA′ being to each other as the sines of the angles which the side of the cone BA′ makes with the two cyclic planes.

PAGE 43, § 6, 2nd Mem.—In his *Histoire de la Geometrie,* p. 236, M. Chasles informs us, that M. Magnus was anticipated in the discovery of the first of these two theorems by Fuss, a Russian geometer, who, in discussing the curve which is the locus of the vertices of all the spherical triangles having the same base and sum of sides, discovered that this curve is the intersection of a sphere with a cone of the second degree, whose vertex is at the centre of the sphere.—(*Nova Acta,* tom. iii. A. D. 1787.) The formulæ employed by Fuss conducted him to the following result, which he calls "*maxime memorabilem,*" viz. that if the sum of the sides be given equal to half the circumference of the sphere, the locus of the vertex will be an arc of a great circle, whatever be the base of the triangle. This is evident from the most elementary considerations of spherical geometry. M. Chasles observes (*Histoire de la Geometrie*, p. 239,) that M. Steiner was the first who proved that the base of a spherical triangle whose area and vertical angle are given, envelopes a spherical conic.

PAGE 46, § 11.—From the two theorems in this article we may deduce the two following:

If two spherical conics have the same cyclic arcs, every arc of a great circle touching the inner curve meets the outer one in two points which are equally distant from the point of contact.

If two spherical conics have the same foci, two arcs drawn from a point in the outer curve to touch the inner one make equal angles with the tangent to the outer conic at the point from which the tangents are drawn.

These last two theorems are only particular cases of the two following:

If two spherical conics have the same cyclic arcs, every arc of a great circle which cuts both of them intersects one curve in two points, which are equally distant from the points in which this arc meets the other curve.

If two spherical conics have the same foci, the angle contained between two arcs touching the two curves will be equal to the angle between the two other tangent arcs which may be drawn to the conics from the same point.

PAGE 47, § 14.—If M. Chasles had not restricted himself to the consideration of the hemisphere on which the conic is traced, (see page 45, § 10,) he might have stated the theorems in this article more elegantly.

For each arc touching the conic meets a cyclic arc in two points diametrically opposite; so that, *if the points of intersection be rightly chosen*, we might assert that

(1.) Two tangent arcs to a spherical conic intersect the two cyclic arcs in four points, which lie in the circumference of a small circle, whose centre is the pole of the great circle passing through the two points of contact of the tangent arcs.

The four vector arcs, drawn from the two foci of a spherical conic to any two points of the curve, will all touch the same small circle, whose centre is the point of concourse of the two arcs touching the curve at these two points.

This mode of stating the preceding theorems is valuable; for from them we are led to those contained in § 16, by the aid of the following propositions, which are very useful in spherical geometry, and may be easily proved:

(2.) If an arc of a great circle, passing through a fixed point on the surface of the sphere, intersect a given small circle in two points, the product of the tangents of the semi-arcs, lying between these two points and the fixed point, will be constant.

If from any point in a fixed arc of a great circle tangent arcs be drawn to a given small circle, the product of the tangents of the semi-angles, which these tangent arcs make with the fixed arc, will be constant.

PAGE 47, § 15.—These theorems might be readily deduced from those in No. 11, by the following method:

Let O be the point of intersection of the cyclic arcs; then, since the portion of a tangent arc AB, intercepted between the two cyclic arcs, is bisected at the point of contact P, the two *consecutive* tangents AB, A′B′, must bisect one another in P; hence, the elementary spherical triangles APA′, BPB′, are equal, and the area of the whole triangle AOB, will remain invariable; the sum of its three angles is therefore constant: but the angle AOB is fixed, therefore, the sum of the angles ABO, BAO, must be constant. Having proved the first theorem thus, we may deduce the second from it by reference to the supplementary cone. Precisely in the same way we might prove the first of the two following theorems, and then derive the second from it by means of the supplementary cone.

If two spherical conics have the same cyclic arcs, any arc touching the inner curve, will cut off from the outer one a segment of a constant area.

If from any point in the outer of two biconfocal conics, two tangent arcs be drawn to the inner curve, the sum of these two arcs and of the concave part of the circumference of the conic included between them will be constant.

PAGE 47, § 17.—By means of the known formula

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s-a)}{\sin b \sin c},$$

expressing the cosine of the half of one angle of a spherical triangle in terms of the three sides, we might at once show that if the base a and the sum of the sides b, c, of a spherical triangle be constant, the product of the sines of the two arcs drawn from the extremities of the base perpendicular to the arc bisecting the supplement of the vertical angle is constant, and equal to

$$\sin^2 \tfrac{1}{2}(b+c) - \sin^2 \tfrac{1}{2} a.$$

PAGE 48, § 21.—It is evident that any arc, passing through a fixed point, and cutting a spherical conic and the polar arc of the fixed point, will be harmonically divided.

PAGE 50, § 28.—If a plane quadrilateral be circumscribed about a circle, the angles which two opposite sides subtend at the centre will be supplemental. Hence we derive the following theorems:

A spherical quadrilateral being circumscribed about a spherical conic, if arcs be drawn from the pole of one of its cyclic arcs through the four vertices of the quadrilateral, so as to meet that cyclic arc, the arcs drawn to two adjacent vertices of the quadrilateral will include between them a portion of the cyclic arc supplemental to that included between the two remaining arcs.

A spherical quadrilateral being inscribed in a spherical conic, if arcs be drawn from one focus to the four points in which the four sides meet the corresponding director arc, the portion of that arc included between two adjacent sides of the quadrilateral will subtend at the focus an angle supplemental to that subtended by the portion of the director arc included between the two other sides.

PAGE 50, § 29.—From the theorems in this article we may deduce two others, which are most powerful instruments in discussing questions of spherical geometry.

It is easy to show that if four arcs diverging from the same point O be cut in the points A, B, C, D, by any fifth arc, we shall always have

$$\frac{\sin AB \sin CD}{\sin AD \sin BC} = \frac{\sin AOB \sin COD}{\sin AOD \sin BOC};$$

or, as M. Chasles designates it, the *anharmonic relation* of the four points is the same as that of the four arcs.

Hence we arrive at the following theorems:

(1.) If from four fixed points on a spherical conic arcs be drawn to any fifth point on the curve, their anharmonic relation will be constant.

If four fixed tangent arcs be drawn to a spherical conic, any fifth tangent arc will cut them in four points, the anharmonic relation of which will be constant.

The first of these theorems, or rather its converse, may often be used with success when we have to investigate the locus described by the vertex of a spherical triangle, two of whose sides pass through two fixed points P, P′, whilst it also fulfils some other conditions. Let C, C′, C″, C‴, be four positions of the vertex; then, if we can show that the anharmonic relation of the arcs PC, PC′, PC″, PC‴, is the same as that of the arcs P′C, P′C′, P′C″, P′C‴, it follows that the locus of C will be a spherical conic passing through the points P, P′.

And in like manner, the second theorem may be employed as advantageously in ascertaining the envelope of the base of a spherical triangle, two of whose angles A, B, are on given arcs L, L′, and which is further limited by some other conditions. Let A, A′, A″, A‴, be four positions of one angle, and B, B′, B″, B‴, the four corresponding positions of the other, then, if we can show that the anharmonic relation of A, A′, A″, A‴, is the same as that of B, B′, B″, B‴, the six arcs AB, A′B′, A″B″, A‴,B‴, L, L′, must all touch the same spherical conic, which will be the required envelope. In what follows we shall give several examples of the application of these principles.

Six points, A, A′, B, B′, C, C′, lying in the arc of a great circle, and corresponding to each other two by two, are said to be *in involution*, when the anharmonic relation of four of them is the same as that of their four conjugates:

For instance, if the anharmonic relation of A, B, C, C′, is the same as that of A′, B′, C′, C, the three couples of points A, A′, B, B′, C, C′, are in involution. And it may be proved that if this relation holds for one set of four points and their conjugates, it will also hold for any other: that is, if the anharmonic relation of A, B, C, C′, is the same as that of A′, B′, C′, C, we shall also have the anharmonic relation of A, B, C, B′, the same as that of A′ B′, C′, B, and so on.

Again, from the definition which has been given for the involution of six points, it may be shown that if there be three or more couples of points such, that the first two couples, taken along with any other couple, form a system in involution, any three of these couples will likewise form a sytem in involution.

Three couples of arcs which pass through the same point are said to be in involution, if the anharmonic relation of any four of them is the same as that of their four conjugates: and such a system possesses properties analagous to those of a system of six points in involution.

Using the preceding definitions and properties of systems of points or arcs in involution, we may deduce the following theorems:

(2.) A spherical quadrilateral being inscribed in a conic, any transversal arc will cut the curve and the four sides of the quadrilateral in six points, which are in involution.

A spherical quadrilateral being circumscribed about a conic, the four arcs drawn to its four vertices from any point without the curve, taken along with the two tangent arcs drawn from the same point, form a system of six arcs in involution.

The theorem in the first column may be thus proved. Let the transversal arc meet two opposite sides of the quadrilateral, *ad*, *bc*, in A, A′, the two other sides *ab*, *cd*, in B, B′, and the conic in C, C′; then, by the first of the theorems (1), the anharmonic relation of the four arcs drawn from *a* to *d*, C, *b*, C′, is the same as that of the arcs drawn from *c* to *b*, C′, *d*, C; therefore the anharmonic relation of the points A, C, B, C′, is the same at that of A′, C′, B′, C; hence the points A, A′, B, B′, C, C′, are in involution.

From the preceding we may successively deduce the following pairs of theorems:

(3.) Two spherical conics being circumscribed about a spherical quadrilateral, any transversal arc of a great circle meets the two curves, and also two opposite sides of the quadrilateral in six points, which are in involution.

Two spherical conics being inscribed in a quadrilateral, four tangent arcs drawn to them from the same point, together with the two arcs drawn from the same point to two opposite vertices of the quadrilateral, form a system of six arcs in involution.

(4.) Three spherical conics being circumscribed about the same quadrilateral, any transversal arc of a great circle meets the three curves in six points, which are in involution.

Three spherical conics being inscribed in the same quadrilateral, the six tangent arcs, drawn to them from any point, will form a system of six arcs in involution.

PAGE 51, § 32.—To the theorems contained in this section we may be permitted to add the following, which are derived from known properties of the circle.

(1) The angle which an arc of a circle subtends at the centre is double of that which it subtends at any point in the remaining part of the circumference. Hence,

If from two fixed points on a spherical conic arcs be drawn through any third point on the curve, they will include between them, on one of the

If any tangent arc be drawn to a spherical conic, intersecting two fixed tangent arcs in two points, and through these points arcs be drawn to one of the cyclic arcs, a portion which will be the half of that included between two arcs drawn from the two fixed points through the pole of the cyclic arc.

foci of the conic, they will contain between them an angle which will be the half of that contained between the two vector arcs drawn to the points of contact of the fixed tangent arcs.

(2.) The portion of a tangent to a circle intercepted between two fixed tangents, subtends a constant angle at the centre. Hence

If two fixed tangent arcs be drawn to a spherical conic, and any third tangent arc be drawn meeting them in two points, the arcs passing through these two points and through the pole of a cyclic arc, will intercept on that cyclic arc a portion of a constant length.

If from two fixed points in a spherical conic arcs be drawn to any third point on the curve, and produced to meet one of the director arcs, they will intercept between them, on that director arc, a portion which will subtend a constant angle at the corresponding focus.

(3.) If from any point in a given right line tangents be drawn to a given circle, they will make with the given right line angles the product of the trigonometric tangents of whose halves is constant. Hence,

If from any point in a fixed arc two arcs be drawn touching a given spherical conic, they will intersect either of its cyclic arcs in two points, such that the product of the trigonometric tangents of the halves of their distances from the point in which the fixed arc meets the given cyclic arc will be constant.

If through a fixed point on the surface of the sphere any arc be drawn meeting a spherical conic in two points, the arcs drawn from either focus to these two points will make, with the arc drawn from the same focus to the given point, two angles, the product of the trigonometric tangents of whose halves will be constant.

(4.) If any right line be drawn through a fixed point in the plane of a given circle, intersecting it in two points, the radii drawn to these points will make angles with the radius passing through the given point such that the product of the trigonometric tangents of their halves will be constant. Hence we derive the two following theorems :

If any arc be drawn through a fixed point, intersecting a given spherical conic in two points, the arcs drawn through these two points and through the fixed point from the pole of one of the cyclic arcs of the conic will meet that cyclic arc in three points, such that the product of the trigonometric tangents of the halves of the distances of the first two from the third will be constant.

If from any point in a fixed arc tangent arcs be drawn to a given spherical conic, and arcs be drawn from one of the foci to the points in which these tangent arcs meet the corresponding director arc, these vector arcs will make, with the vector arc drawn to the point in which the fixed arc meets the director arc, two angles, the product of the trigonometric tangents of whose halves will be constant.

(5.) A plane triangle being inscribed in a circle, if from any point P in the circumference right lines Pa, Pb, Pc, be drawn, respectively meeting the three sides BC, CA, AB, and making with them equal angles on the same side of the lines Pa, Pb, Pc, then the three points, a, b, c, will lie in the same straight line. Hence,

If the three sides, a, b, c, of a spherical triangle inscribed in a conic, be successively produced to meet one of the cyclic arcs in the points α, β, γ, and equal portions $\alpha\lambda$, $\beta\mu$, $\gamma\nu$, be measured on the cyclic arc in the same direction, arcs drawn from the points λ, μ, ν, through any point on the curve will meet the sides a, b, c, in three points lying in the arc of a great circle.

From the three vertices of a spherical triangle ABC circumscribed about a conic, if arcs be drawn to one of the foci F, and three other vector arcs Fl, Fm, Fn, be drawn so that the angles AFl, BFm, CFn, may be equal and lie at the same side of the arcs FA, FB, FC; these three arcs, Fl, Fm, Fn, will meet any tangent arc in three points l, m, n, such that the arcs Al, Bm, Cn, will pass through the same point.

PAGE 51, § 33.—Two right lines and a fixed point being given, if a constant angle be made to turn round this point as vertex, the right line joining the points in which its sides meet the two given right lines will envelope a conic section (p. 62, iii. 1). Hence,

If a variable spherical angle turn round a fixed point on the surface of a sphere so as to intercept between its sides a constant segment on a given arc, the arc joining the points in which its sides meet two other fixed arcs will envelope a spherical conic touching those two fixed arcs.

If a constant spherical angle turn round a given point as vertex, the arcs joining the points in which its sides meet a fixed arc with two other fixed points will intersect in a point, the locus of which will be a spherical conic passing through those two fixed points.

PAGE 53, § 38.—The theorems contained in this article might be stated more generally as follows:

If two arcs be made to turn round two fixed points on a spherical conic, so as to intersect in any third point of the curve, the arc joining the points in which they respectively meet two fixed arcs will envelope a spherical conic touching these two fixed arcs.

Two fixed tangent arcs being drawn to a spherical conic, any third tangent arc will intersect them in two points, and the arcs respectively drawn through these points and through two fixed points will intersect in a point the locus of which will be a spherical conic passing through the two fixed points.

The principles stated in page 78 furnish us with easy proofs of these theorems; that in the first column may be proved thus: Let P, P′, be the two fixed points; C, C′, C″, C‴, four other points on the curve; then, since the anharmonic function of the arcs PC, PC′, PC″, PC‴, is the same as that of P′C, P′C′, P′C″, P′C‴, the anharmonic function of the four points A, A′, A″, A‴, in which PC, PC′, PC″, PC‴, intersect one given arc, will be the same as that of the four points B, B′, B″, B‴, in which P′C, P′C′, P′C″, P′C‴, meet the other given arc; therefore, the arcs AB, A′B′, A″B″, A‴B‴, are tangents to a spherical conic touching the two given arcs.

The theorem in the second column might be similarly proved, or we may infer it from the first by reference to the supplementary cone.

PAGE 53, § 39.—Since the arc which bisects any angle of a spherical triangle divides the opposite side into segments, the sines of which are proportional to the sines of the conterminous sides, we infer from the theorem in this article, that

If the base of a spherical triangle be given, and also the ratio of the sines of the two remaining sides, the locus of the vertex will be a spherical conic, whose cyclic arcs will be in the two planes perpendicular to the radii of the sphere drawn to the points which divide the given base internally and externally, so that the sines of the segments may be in the given ratio.

If the vertical angle of a spherical triangle be given, and also the ratio of the sines of the two remaining angles, the base will envelope a spherical conic, whose foci will be the extremities of the radii of the sphere perpendicular to the planes of the two arcs which divide the given vertical angle internally and externally, so that the sines of the segments may be in the given ratio.

PAGE 53, § 40.—The theorem in the second column shows that if the base of a spherical triangle be given, and also the product of the cosines of the two remaining sides, the locus of the vertex will be a spherical conic.

PAGE 53, § 42.—To the theorems given by M. Chasles in this section, we may add the following:

(1). The chord in a circle joining the extremities of two radii which contain between them a constant angle envelopes a circle concentric with the given one. From this property of the circle we derive the following theorems relative to spherical conics.

A spherical conic and one of its cyclic arcs being given, if round the pole of this cyclic arc, as vertex, a spherical angle of variable magnitude be made to turn, whose sides intercept between them on the cyclic arc a portion of a constant length, the arc joining the points in which the sides of the moveable angle meet the given conic will envelope a second conic.

The given cyclic arc will be a cyclic arc of the new conic; and this arc will have the same pole with relation to the two curves.

A spherical conic and one of its foci being given, if round that focus, as vertex, a constant spherical angle be made to turn, and from the points in which the sides of this angle meet the director arc corresponding to the given focus two tangent arcs be drawn to the given conic, their point of concourse will generate a second spherical conic.

The given focus will be a focus of the new conic; and the corresponding director arc will be the same for the two curves.

(2). If right lines be drawn from two fixed points in the circumference of a circle through the extremities of any diameter, they will intersect in a point the locus of which will be circle, which passes through the two fixed points, and whose centre is the pole of the right line joining them. Hence we derive the following theorems.

A spherical conic and one of its cyclic arcs being given, if arcs be drawn from two fixed points on the curve to the extremities of any arc passing through the pole of that cyclic arc, and terminated by the curve, they will intersect in a point the locus of which will be a second spherical conic.

The given cyclic arc will be a cyclic arc of the new conic, and its pole, with relation to that curve, will be the same as the pole, with relation to the given conic, of the arc joining the two fixed points.

A spherical conic and one of its foci being given, if tangent arcs be drawn to the curve from any point in the corresponding director arc, the arc joining the points in which these tangent arcs meet two fixed tangent arcs will envelope a second spherical conic.

The given focus will be one of the foci of the new conic, and its corresponding director arc, for that curve, will be the arc joining the points of contact of the two fixed tangent arcs.

(3). The principles stated in page 78 furnish us with easy proofs of the following theorems:

In a spherical triangle, if the base and the difference of the base angles be given, the locus of the vertex will be a spherical conic, passing through the extremities of the given base.

In a spherical triangle, if the vertical angle and the difference of the sides containing it be given, the base will envelope a spherical conic touching the two sides which contain the given angle.

(4). If two tangents to a parabola intersect at a constant angle, the radii vectores drawn from the focus to the two points of contact will also contain between them a constant angle. But in any conic section the point of concourse of the tangents at the extremities of two focal radii vectores, which contain between them a constant angle, will generate a conic section, (see page 62, iii. 2). Hence we derive the following very general properties of spherical conics:

If two tangent arcs to a given spherical conic intercept between them a segment of a constant length on a fixed tangent arc to the curve, their point of concourse will generate a second spherical conic.

If the segment intercepted on the fixed tangent be a quadrant, the point of concourse of the two tangent arcs will move along an arc of a great circle.

If a constant spherical angle turn round a fixed point on a given conic, as vertex, the arc joining the points in which its sides meet the curve will envelope a second spherical conic.

If the constant spherical angle be a right angle, the arc which it subtends in the spherical conic will pass through a fixed point.

(5). From the theorem in the second column it appears that if a constant angle turn round a fixed point on a conic section as vertex, the chord which it subtends will envelope a second conic section. Hence we deduce the following theorems:

If a variable spherical angle turn round a fixed point on a spherical conic so that the segment intercepted between its sides on a fixed arc may be of a constant length, the arc joining the

If a constant spherical angle turn round a fixed point as vertex, and if from the points in which its sides meet a fixed tangent arc to a given spherical conic two arcs be drawn touching the points in which these sides meet the conic will envelope a second spherical conic.

curve, their point of concourse will generate a second spherical conic.

(6). Again, from the second of these latter theorems we learn, that if a constant angle turn round a fixed point in the plane of a conic section as vertex, and if from the points in which its sides meet a fixed tangent two other tangents be drawn to the curve, their point of concourse will generate a second conic section. From this we deduce the following theorems:

If a variable spherical angle turn round a fixed point on the surface of a sphere so that the segment intercepted between its sides on a fixed arc may be of a given length, and if, from the points in which its sides meet a fixed tangent arc to a given spherical conic, two other tangent arcs be drawn to the curve, their point of concourse will generate a second spherical conic.

If a constant spherical angle turn round a fixed point on the surface of a sphere, and, from the points in which its sides meet a fixed arc, two arcs be drawn to a fixed point on a given spherical conic, the arc joining the points in which these two arcs meet the curve will envelope a second spherical conic.

As before, we might, from the theorem in the second column, deduce a property of the plane conic sections, and from it in turn derive a pair of theorems relating to spherical conics. In fact there is no limit to the number of theorems which might be obtained in this way.

Page 58, § 56.—The following theorems, which are more general than those given by M. Chasles in this article, may be proved by means of the principles laid down in page 78.

If any two spherical angles, each of invariable magnitude, turn round two fixed points as vertices, so that two of their sides intersect on a given spherical conic passing through the two fixed points, the point of intersection of their two other sides will generate a spherical conic, which will pass through the two fixed vertices of the moveable angles.

If along two fixed arcs any two segments, each of invariable magnitude, be made to move, so that the arc joining two of their extremities may be a tangent to a given spherical conic which touches the two fixed arcs, the arc joining their two other extremities will envelope a spherical conic which will touch the two fixed arcs along which the segments move.

The first theorem may be proved in the following manner:

Let P, P′, be the two fixed points round which the constant angles, MPC, MP′C, turn; let M, M′, M″, M‴, be four points on the given conic, and C, C′, C″, C‴, four positions of the point the locus of which is sought. Then, since the angles MPC, M′PC′, M″PC″, M‴PC‴, are equal, the anharmonic relation of the arcs PM, PM′, PM″, PM‴, is the same as that of the four arcs, PC, PC′, PC″, PC‴; and since the angles MP′C, M′P′C′, M″P′C″, M‴P′C‴, are equal, the anharmonic relation of the four arcs, P′M, P′M′, P′M″, P′M‴, is the same as that of the four arcs, P′C, P′C′, P′C″, P′C‴. But by the first theorem given in the note to page 50, § 29, the anharmonic relation of PM, PM′, PM″, PM‴, is the same as that of P′M, P′M′, P′M″, P′M‴; hence the anharmonic relations of the two systems of arcs PC, PC′, PC″, PC‴, and P′C, P′C′, P′C″, P′C‴, are the same: consequently a conic section will pass through the points P, P′, C, C′, C″, C‴.

The second theorem might be proved in a similar manner, or we may derive it from the first by means of the supplementary cone.

PAGE 66, cap. viii.—The theorems contained in this chapter are deduced by M. Chasles from preceding ones by means of that particular polar transformation in which a parabola is used as the auxiliary conic.—(See *Quetelet's Correspondance Mathematique et Physique*, Tom. V. p. 281.)

We may, however, arrive at them without having recourse to this method.

The first follows immediately from the theorem given in the note to page 51, § 33.

The second and third are particular cases of the theorems relative to the plane conic sections which may be derived from the theorems (5) and (6) first column, in the note to page 53, § 42.

PAGE 68, cap. ix. § 5.—This theorem, which M. Chasles obtains by a polar transformation, follows directly from that given in page 63, cap. iv. § 6.

PAGE 69, cap. x. § 5.—The theorem which M. Chasles here gives for the construction of the conic sections by points admits of a great extension.

If two spherical angles, which intercept segments of constant lengths on two fixed arcs, turn rourd two fixed points as vertices, so that two of their sides always intersect upon a given spherical conic passing through the two fixed points, the point of concourse of their two other sides will generate a second spherical conic passing through the two fixed points.

This may be readily proved. For let P, P′, be the two fixed points, and MPC, MP′C, the two constant angles: let M, M′, M″, M‴, be four points on the given conic, and C, C′, C″, C‴, four positions of the point whose locus is sought, then the anharmonic relations of the two systems of arcs, PM, PM′, PM″, PM‴, and PC, PC′, PC″, PC‴, must be the same; since the angles MPC, M′PC′, M″PC″, M‴PC‴, intercept equal segments on the same arc. For a similar reason, the anharmonic relations of the two systems P′M, P′M′, P′M″, P′M‴, and P′C, P′C′, P′C″, P′C‴, must be the same: but since the systems PM, PM′, PM″, PM‴, and P′M, P′M′, P′M″, P′M‴, have the same anharmonic relation, it follows that the two systems PC, PC′, PC″, PC‴, and P′C, P′C′, P′C″, P′C‴, must also have the same anharmonic relation: so that a spherical conic must pass through the six points P, P′, C, C′, C″, C‴.

PAGE 70, cap. x. § 6.—This last theorem may also be extended.

If two spherical angles of constant magnitudes be made to turn round two fixed points, so that the arc joining the points in which two of their sides respectively meet two fixed arcs, may always be a tangent to a given spherical conic touching those two fixed arcs, the arc joining the points in which the two other sides respectively meet the same two arcs will envelope a second spherical conic, which will also touch these two fixed arcs.

For let P, P′, be the two fixed points, round which the constant angles APα, BP′β, turn; Aα, Bβ, being the two fixed arcs. Let A′, A″, A‴, be three other positions of the point A, and α′, α″, α‴, B′, B″, B‴, β′, β″, β‴, the corresponding positions of the points α, B, β; then the anharmonic relations of the two systems of points A, A′, A″, A‴, and α, α′, α″, α‴, are the same: likewise the anharmonic relations of B, B′, B″, B‴, and β, β′, β″, β‴, are the same: but if AB, A′B′, A″B″, A‴B‴, be tangents to the same conic which touches the two arcs Aα, and Bβ, the anharmonic relations of the two systems A, A′, A″, A‴, and B, B′, B″, B‴, will be the same, and therefore α, α′, α″, α‴, and β, β′, β″, β‴, will be similar systems; consequently the arcs αβ, α′β′, α″β″, α‴β‴, must all be tangents to aspherical conic which touches the two arcs Aα, Bβ.

The theorem just proved might have been deduced from the one contained in the preceding note, by employing two supplementary cones.

APPENDIX.

ON THE APPLICATION OF ANALYSIS TO SPHERICAL GEOMETRY.

§ 1.—*On the Use of Spherical Coordinates.*

As the position of a point on a plane is determined by reference to two fixed right lines, or axes of coordinates, in that plane, so the position of a point on the surface of a sphere may be determined by referring it to two fixed arcs of great circles.

Through a point O on the surface of a sphere, which we shall call *the origin*, let two fixed arcs of great circles OX, OY, be drawn, and let the points X and Y be 90° distant from O; then if arcs be drawn from Y and X through any point P on the sphere, and respectively meeting OX and OY in M and N, the trigonometric tangents of the arcs OM, ON, are to be considered as the *spherical coordinates* of the point P, and we shall denote them by x and y. To the fixed arcs I propose to give the name of *arcs of reference.*

If the arcs OX, OY, XN, YM, be projected, by means of radii of the sphere, into right lines upon a tangent plane at the point O, the projections of the arcs XN, YM, will be respectively parallel to the projections of OX and OY; consequently, the projections of the arcs OM, ON, which are the trigonometric tangents of those arcs, or the spherical coordinates of the point P, are the rectilinear coordinates, in the tangent plane, of the projection of the point P; the axes of coordinates being the projections of the arcs of reference. This consideration leads to an important consequence, viz.: that *an equation of the n^{th} degree between the spherical coordinates x and y represents a curve formed by the intersection of the sphere with a cone of the n^{th} degree having its vertex at the centre of the sphere.*

Each side of such a cone would meet the surface of the sphere in two points diametrically opposite, which, however, will have the same spherical coordinates since $\tan\theta = \tan(180° + \theta)$.

Thus the equation of the first degree between x and y represents an arc of a great circle. An equation of the second degree represents a spherical conic, and so on.

In what follows, for the sake of simplicity, I shall suppose the arcs of reference to be at right angles to each other. In this case the arcs PM, PN, are perpendicular to OX and OY, and I call such a system of spherical coordinates *rectangular.*

§ 2.—*The Equation of an Arc of a great Circle.*

The equation of the first degree between x and y may in general be written in the form

$$\alpha x + \beta y = 1. \qquad (1)$$

The great circle, represented by this equation, meets the arcs of reference in two points, the cotangents of whose distances from the origin are, α and β. Hence, denoting the coordinates of the pole of this great circle by x' and y', we shall have

$$x' = -\alpha \text{ and } y' = -\beta. \qquad (2)$$

It appears from this that if α and β, instead of being fixed, are merely connected by an equation of the first degree, the great circle (1) will turn round a fixed point. If the equation connecting α and β be of the second degree, the great circle will envelope a spherical conic. And, in general, if α and β be connected by an equation of the n^{th} degree, the great circle will envelope a spherical curve to which n tangent arcs may be drawn from a point without it. For the pole of this great circle will generate a curve formed by the intersection of the sphere with a cone of the n^{th} degree, and as a transversal arc of a great circle may meet this curve in n points, the great circle (1) may assume n different positions whilst it passes through the same point.

But the general equation of the first degree may be written in the form

$$y = mx + n \qquad (3)$$

and it is desirable to explain the geometric meaning of the constants m and n. As to n, it is evidently the coordinate of the point in which the great circle (3) meets the y arc of reference.

If we take another great circle whose equation is $y = mx$, and seek the coordinates of the point in which this great circle meets the circle (3), we should find infinite values for x and y. These two great circles will therefore intersect on the great circle of which the origin is the pole: and, consequently, in the equations of two arcs of great circles $y = mx + n$, $y = m'x + n'$, if $m = m'$, these two arcs will pass through the same point in the great circle of which the origin is the pole. Another geometric meaning may be given to the constant m: for α and β are evidently proportional to the cosines of the angles which the arc (1) makes with the arcs of reference, and $m = -\frac{\alpha}{\beta}$.

The equation of an arc of a great circle passing through a fixed point x', y', is

$$y - y' = m(x - x') \qquad (4)$$

where m is indeterminate; and the equation of the great circle passing through two given points x', y', and x'', y'', is

$$\frac{y-y'}{x-x'}=\frac{y''-y'}{x''-x'} \tag{5}$$

§ 3. *To express the Distance between two Points on the Sphere in Terms of their Coordinates x', y', and x'', y''.*

Let δ be the distance between the points; ϱ' and ϱ'' their distances from the origin; and ω', ω'', the angles which ϱ' and ϱ'' make with the x arc of reference. Then we shall have the following equation

$$\cos\delta=\cos\varrho'\cos\varrho''+\sin\varrho'\sin\varrho''\cos(\omega''-\omega'),$$

but we also have

$$\begin{array}{ll}\cos\omega'=x'\cot\varrho' & \cos\omega''=x''\cot\varrho''\\ \sin\omega'=y'\cot\varrho' & \sin\omega''=y''\cot\varrho''\end{array}$$

and by squaring these last equations and adding them two by two we shall find

$$\cos\varrho'=\frac{1}{\pm\sqrt{1+x'^2+y'^2}}\text{ and }\cos\varrho''=\frac{1}{\pm\sqrt{1+x''^2+y''^2}}.$$

Substituting from the last six equations in the preceding one we obtain,

$$\cos\delta=\frac{1+x'x''+y'y''}{\pm\sqrt{(1+x'^2+y'^2)(1+x''^2+y''^2)}}. \tag{6}$$

From this we get the following formulæ which are more generally useful,

$$\sin\delta=\pm\sqrt{\frac{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2}{(1+x'^2+y'^2)(1+x''^2+y''^2)}} \tag{7}$$

$$\tan\delta=\frac{\pm\sqrt{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2}}{1+x'x''+y'y''}. \tag{8}$$

The last three formulæ have double signs, because they give the distance of the point x', y', from the point diametrically opposite to x'', y'', as well as from the point x'', y'' itself. And it must also be remembered that two points on the surface of a sphere may be joined by two different arcs of great circles, which taken together make up the whole circumference of a great circle.

§ 4.—*To express the Length of the Arc λ, drawn from a given Point x', y', at right Angles to the Arc of a great Circle whose Equation is $\alpha x+\beta y=1$.*

The required arc is the complement of the distance of the given point from the pole of the given circle, whose coordinates are $-\alpha$, and $-\beta$; substituting these values for x'', y'' in (6) we find,

$$\sin\lambda = \frac{1 - \alpha x' - \alpha y'}{\pm\sqrt{(1+\alpha^2+\beta^2)(1+x'^2+y'^2)}}. \qquad (9)$$

§ 5.—*To express the Angle θ between two Arcs whose Equations are $\alpha x + \beta y = 1$ and $\alpha' x + \beta' y = 1$.*

Since the angle between the arcs is equal to the distance between their poles, we find from (6)

$$\cos\theta = \frac{1 + \alpha\alpha' + \beta\beta'}{\pm\sqrt{(1+\alpha^2+\beta^2)(1+\alpha'^2+\beta'^2)}}. \qquad (10)$$

This formula shows that the two arcs will be at right angles to each other if

$$1 + \alpha\alpha' + \beta\beta' = 0. \qquad (11)$$

§ 6.—*To find the Equation of the great Circle passing through the Point x', y', and perpendicular to the Arc whose Equation is $\alpha x + \beta y = 1$.*

The great circle whose equation is required must pass through the pole of the given great circle, as well as through the point x', y'; hence from (5) it appears that its equation will be

$$\frac{y - y'}{x - x'} = \frac{\beta + y'}{\alpha + x'}. \qquad (12)$$

§ 7.—*Transformation of spherical Coordinates.*

Spherical coordinates may be transformed by assuming a new origin and new arcs of reference. It is not, however, very easy to establish the general formulæ which enable us to pass from any one system of spherical coordinates to another. I have investigated these formulæ and will furnish them on another occasion, having at present no need to employ any but the simplest modes of transformation. In the first place, the formulæ which are to be used, in passing from one system of rectangular coordinates to another, whilst the origin remains the same, are

$$\begin{aligned} x &= x' \cos\alpha - y' \sin\alpha \\ y &= y' \cos\alpha + x' \sin\alpha \end{aligned} \qquad (13)$$

α being the angle between the given and the new reference arcs of x. This becomes evident if we project the two systems of arcs of reference into right lines upon the tangent plane at the common origin. Next, we may change the origin, whilst the reference arc of x remains the same; the new system of coordinates as well as the given one being rectangular. The formulæ for this transformation are

$$x = \frac{x' + \tan\alpha}{1 - x'\tan\alpha} \quad y = \frac{y' \sec\alpha}{1 - x'\tan\alpha}. \qquad (14)$$

where α denotes the distance between the two origins.

To prove this, let OX, OY, be the given arcs of reference, and O′X, O′Y, the new ones; and from any point P draw arcs PM, PN, PN′, respectively perpendicular to OX, OY, O′Y; then, as the arc PM passes through Y

$$\frac{\tan \text{YN}'}{\tan \text{YN}} = \frac{\cos \text{PYN}'}{\cos \text{PYN}} = \frac{\cos \text{O}'\text{M}}{\cos (\text{OO}' + \text{O}'\text{M})};$$

hence, since cot YN, cot YN′ are respectively equal to y and y' $\text{OO}' = \alpha$, and $\tan \text{O}'\text{M} = x'$, we get

$$y = \frac{y' \sec \alpha}{1 - x' \tan \alpha};$$

also, since $\text{OM} = \text{OO}' + \text{O}'\text{M}$,

$$\tan \text{OM} = \frac{\tan \text{OO}' + \tan \text{O}'\text{M}}{1 - \tan \text{OO}' \tan \text{O}'\text{M}};$$

therefore

$$x = \frac{x' + \tan \alpha}{1 - x' \tan \alpha}$$

In like manner, if we wished to transfer the origin to a point on the y arc of reference, that arc being still retained, and the new system of coordinates being rectangular, we should have to use the formulæ,

$$x = \frac{x' \sec \beta}{1 - y' \tan \beta}, \quad y = \frac{y' + \tan \beta}{1 - y' \tan \beta}, \tag{15}$$

where β denotes the distance between the origins.

§ 8.—*A Curve on the Surface of the Sphere being represented by an Equation between spherical Coordinates, to determine the Equation of the great Circle touching it at a given Point x', y'.*

The equation of a great circle, passing through the points x', y', and x'', y'', on the curve, is

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'};$$

and if x'', y'' approach indefinitely near to x', y', this equation becomes

$$\frac{y - y'}{x - x'} = \frac{dy'}{dx'}; \tag{16}$$

This is the equation of the tangent arc at the point x', y': we may put it into the following symmetrical form:

$$\frac{x dy'}{x' dy' - y' dx'} + \frac{y dx'}{y' dx' - x' dy'} = 1. \tag{17}$$

The formulæ (16) and (17) will evidently hold good whether the arcs of reference be rectangular or not.

§ 9.—*To determine the Equation of the normal Arc to a spherical Curve at a given Point* x', y'.

It appears from (12) that the equation of the arc passing through the point x', y', and perpendicular to the arc whose equation is (17) will be

$$(y-y')\,[dy'+x'\,(x'dy'-y'dx')]+(x-x')[dx'+y'\,(y'dx'-x'dy')]=0 \quad (18)$$

This is the equation of the normal arc at the point x', y'. The coordinates in this last formula are supposed to be rectangular.

§ 10.—The *spherical evolute* of a curve described on the surface of the sphere is the locus of the point in which two consecutive normal arcs to the curve intersect. In finding the spherical evolute, we may follow a course exactly analogous to that by which we investigate the evolutes of plane curves. We must take the equation of the normal arc (18) and differentiate it with respect to x' and y': supposing x and y to be constant; the resulting equation may be written in the form

$$\left.\begin{array}{l}(y-y')\,[d^2y'+x'\,(x'd^2y'-y'd^2x')+(x'dy'-y'dx')dx'] \\ +(x-x')\,[d^2x'+y'\,(y'd^2x'-x'd^2y')+(y'dx'-x'dy')\,dy'] \\ \qquad =dx'^2+dy'^2+(x'dy'-y'dx')^2\end{array}\right\}(19)$$

From this and the equation of the normal arc (18) we find

$$\left.\begin{array}{c} y-y'= \\ \dfrac{[dx'^2+dy'^2+(x'dy'-y'dx')^2]\,[dx'+y'\,(y'dx'-x'dy')]}{(dx'd^2y'-dy'd^2x')\,(1+x'^2+y'^2)+(x'dy'-y'dx')\,(dx'^2+dy'^2+(x'dy'-y'dx')^2)} \\ x-x'= \\ \dfrac{[dx'^2+dy'^2+(x'dy'-y'dx'^2]\,[dy'+x'(x'dy'-y'dx')]}{(dy'd^2x'-dx'd^2y')(1+x'^2+y'^2)+(y'dx'-x'dy')(dx'^2+dy'^2+(x'dy'-y'dx')^2)}\end{array}\right\}(20)$$

By these last equations taken along with that of the given curve and its first and second differentials, we may eliminate x', y', dx', dy', d^2x', d^2y'; and the resulting equation between x and y will be the required equation of the spherical evolute.

§ 11.—*To determine the Circle of the Sphere which osculates a spherical Curve at a given Point* x', y'.

The spherical coordinates of the centre of the osculating circle are evidently the x and y in formulæ (20). To ascertain the magnitude of this circle we must have recourse to the formula (8), from which we find the tangent of the arc joining the points x, y, and x', y'. In the formulæ (20) let us put for shortness,

$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2 = \text{A},$$

$$dx'd^2y' - dy'd^2x' = \text{B},$$

$$\text{B}(1 + x'^2 + y'^2) + (x'dy' - y'dx')\,\text{A} = \text{C},$$

then we shall find from them

$$y = \frac{\text{B}(1 + x'^2 + y'^2)\,y' + \text{A}dx'}{\text{C}} \quad x = \frac{\text{B}(1 + x'^2 + y'^2)\,x' - \text{A}dy'}{\text{C}} \quad (21)$$

and

$$yx' - xy' = \frac{\text{A}\,(x'dx' + y'dy')}{\text{C}}.$$

Hence,

$$\text{C}\,(1 + xx' + yy') = \text{C} - \text{A}(x'dy' - y'dx') + \text{B}\,(1 + x'^2 + y'^2)(x'^2 + y'^2)$$
$$= \text{B}\,(1 + x'^2 + y'^2)^2.$$

Also,

$$\text{C}^2\,[(x' - x)^2 + (y' - y)^2 + (xy' - yx')^2] = \text{A}^3\,(1 + x'^2 + y'^2).$$

Therefore, if γ be the arc joining the points x, y, and x', y',

$$\left.\begin{aligned} \tan\gamma &= \pm\frac{\text{A}^{\frac{3}{2}}}{\text{B}\,(1 + x'^2 + y'^2)^{\frac{3}{2}}} \\ &= \pm\frac{[dx'^2 + dy'^2 + (x'dy' - y'dx')^2]^{\frac{3}{2}}}{(1 + x'^2 + y'^2)^{\frac{3}{2}}\,(dx'd^2y' - dy'd^2x')}. \end{aligned}\right\}(22)$$

This remarkable formula, exactly analogous to that which expresses the radius of the circle osculating a given plane curve at the point x', y', might be obtained from the equation of a circle of the sphere by the following method. It appears from formula (6) that the equation of a circle of the sphere, whose spherical radius is γ, and the coordinates of whose centre are x'', y'', is

$$\cos\gamma = \frac{1 + xx'' + yy''}{\sqrt{(1 + x''^2 + y''^2)\,(1 + x^2 + y^2)}},$$

now this equation may be written in the form

$$ax + by + c = \sqrt{1 + x^2 + y^2}, \qquad (23)$$

where

$$a^2 + b^2 + c^2 = \sec^2\gamma,$$

and in order that the circle, whose equation is (23) may osculate a given spherical curve at the point x', y', we must be able to change x and y, into x' and y', both in the equation (23) and in its first and second differentials. Thus, a, b, c, are completely determined from the three equations

$$ax' + by' + c = \sqrt{1 + x'^2 + y'^2},$$

$$adx' + bdy' = \frac{x'dx' + y'dy'}{\sqrt{1 + x'^2 + y'^2}}$$

$$ad^2x' + bd^2y' =$$
$$\frac{(x'd^2x' + y'd^2y')(1 + x'^2 + y'^2) + dx'^2 + dy'^2 + (x'dy' - y'dx')^2}{(1 + x'^2 + y'^2)^{\frac{3}{2}}}.$$

Hence, using the abbreviations employed in the preceding investigation, we obtain

$$a = \frac{\mathrm{B}(1 + x'^2 + y'^2)\,x' - \mathrm{A}dy'}{\mathrm{B}(1 + x'^2 + y'^2)^{\frac{3}{2}}}$$

$$b = \frac{\mathrm{B}(1 + x'^2 + y'^2)\,y' + \mathrm{A}dx'}{\mathrm{B}(1 + x'^2 + y'^2)^{\frac{3}{2}}},$$

$$c = \frac{\mathrm{B}(1 + x'^2 + y'^2) + \mathrm{A}(x'dy' - y'dx')}{\mathrm{B}(1 + x'^2 + y'^2)^{\frac{3}{2}}},$$

and these three equations give us, since

$$a^2 + b^2 + c^2 - 1 = \tan^2\gamma,$$

$$\tan^2\gamma = \frac{\mathrm{A}^2}{\mathrm{B}^2(1 + x'^2 + y'^2)^3}.$$

§ 12.—*To find the differential of the Arc of a spherical Curve.* The equation (7) may be put into the form

$$\sin\delta = \pm\sqrt{\frac{(x''-x')^2 + (y''-y')^2 + (x'(y''-y') - y'(x''-x'))^2}{(1 + x'^2 + y'^2)(1 + x''^2 + y''^2)}};$$

therefore, if we use ds to denote the differential of the arc of a spherical curve, we shall have

$$ds = \frac{\sqrt{dx'^2 + dy'^2 + (x'dy' - y'dx')^2}}{1 + x'^2 + y'^2}. \qquad (24)$$

This expression enables us to present the equation (22), which gives the tangent of the radius of the osculating circle, in a new form,

$$\tan\gamma = \pm\frac{ds^3(1 + x'^2 + y'^2)^{\frac{3}{2}}}{dx'd^2y' - dy'd^2x'}. \qquad (25)$$

§ 13.—*On the Use of polar Coordinates in spherical Geometry.* In the analytic geometry of the plane, polar coordinates may sometimes be employed more advantageously than rectilinear ones: so, it is sometimes most convenient to express the position of a point on the surface of the sphere by means of the vector arc SP drawn to it from a fixed point S, and the angle PSX between this vector arc and a fixed arc passing through S.

The following formulæ, in which ϱ denotes the arc SP, and ω the angle PSX, are those most commonly used in applications of the method of spherical polar coordinates.

A curve being represented by an equation between ϱ and ω, it may be required to find the angle θ between the vector arc SP and an arc of a great circle touching the curve at the point P. This may be readily done. For, let a vector arc be drawn to another point Q, on the curve infinitely near to P; and from Q let an arc QT be drawn perpendicular to SP: then

$$\tan \text{QPT} = \frac{\text{QT}}{\text{PT}}.$$

But ultimately QPT $= \theta$, QT $= \sin \varrho \, d\omega$, and PT $= d\varrho$, hence we obtain the desired formulæ

$$\tan \theta = \frac{\sin \varrho \, d\omega}{d\varrho}. \qquad (26)$$

Since QP is ultimately the differential of the arc ds, we find

$$ds = \pm \sqrt{\sin^2 \varrho \, d\omega^2 + d\varrho^2}. \qquad (27)$$

and

$$\sin \theta = \frac{\sin \varrho \, d\omega}{ds}. \qquad (28)$$

From (28) we obtain an expression for the arc p, drawn from S perpendicular to the tangent arc at P.

$$\sin p = \frac{\sin^2 \varrho \, d\omega}{\pm \sqrt{\sin^2 \varrho \, d\omega^2 + d\varrho^2}}. \qquad (29)$$

Next, we may find the area of the elementary spherical triangle whose base is QP, the differential of the arc, and whose vertex is the pole S. The known formula

$$\tan \tfrac{1}{2} (\text{area}) = \frac{\tan \frac{1}{2} a \, \tan \frac{1}{2} b \, \sin \text{C}}{1 + \tan \frac{1}{2} a \, \tan \frac{1}{2} b \, \cos \text{C}},$$

which gives the area of a spherical triangle in terms of two sides and the included angle, becomes, when we substitute ϱ for a and b, and $d\omega$ for c,

$$\text{area} = 2 \sin^2 \tfrac{1}{2} \varrho \, d\omega = (1 - \cos \varrho) \, d\omega. \qquad (30)$$

An elegant expression may be obtained for the radius of the osculating circle in terms of ϱ and p. Let C be the centre of this circle, let γ be its radius, and let the arc SC be denoted by δ: then, using the notation already established, we shall have

$$\cos \delta = \cos \varrho \cos \gamma + \sin \varrho \sin \gamma \sin \theta,$$

or,

$$\cos \delta = \cos \varrho \cos \gamma + \sin p \sin \gamma,$$

and this equation may be differentiated on the supposition that δ and γ remain constant. Thus we obtain

$$\tan \gamma = \frac{\sin \varrho \, d\varrho}{\cos p \, dp}. \qquad (31)$$

If we differentiate (29), considering ω as the independent variable, we shall find $\cos p \, dp$ in terms of ϱ and ω; so that we arrive finally at the following equation which gives the tangent of the radius of the osculating circle in terms of ϱ and ω,

$$\tan \gamma = \frac{\pm (\sin^2 \varrho \, d\omega^2 + d\varrho^2)^{\frac{3}{2}}}{(\sin^2 \varrho \cos \varrho \, d\omega^2 + 2 \cos \varrho \, d\varrho^2 - \sin \varrho \, d^2 \varrho) \, d\omega}. \qquad (32)$$

§ 14.—*Mode of passing from rectangular to polar spherical Coordinates.*

To effect this change we have only to put

$$x = \tan \varrho \cos \omega, \text{ and } y = \tan \varrho \sin \omega. \qquad (33)$$

From these equations we may deduce the following formula which is often useful,

$$d\omega = \frac{x dy - y dx}{x^2 + y^2}. \qquad (34)$$

§ 15.—*Quadrature of a spherical Curve given by an Equation between rectangular spherical Coordinates.*

We may derive from (34) and (30) a formula for the quadrature of a spherical curve, which is given by an equation between rectangular spherical coordinates.

$$\text{area} = \int \frac{x dy - y dx}{x^2 + y^2} - \int \frac{x dy - y dx}{(x^2 + y^2) \sqrt{1 + x^2 + y^2}}. \qquad (35)$$

The portion of the sphere whose area is determined by this last formula is included by an arc of the curve and two arcs of great circles drawn to its extremities from the origin. But we may derive from (30) a formula for the quadrature of spherical curves more analagous to that which is ordinarily employed in the quadrature of plane curves.

Let PM, P′M′, be two arcs drawn perpendicular to OX from two consecutive points on the curve. It is to be observed that the letters here employed refer to the construction indicated in § 1, Then the area of the elementary spherical quadrilateral PMM′P′, which we shall consider as the differential of the area of the curve described by P, may be easily found. For, Y being the pole of OX, since the area of the elementary spherical triangle PYP′ is by (30) equal to

$$(1-\cos \text{PY})\, \text{MM}' = (1-\sin \text{PM})\, \text{MM}',$$

it appears that the area of

$$\text{PMM}'\text{P}' = \sin \text{PM}.\text{MM}'.$$

Now

$$\sin \text{PM} = \sin \varrho \sin \omega = \frac{y}{\sqrt{1+x^2+y^2}},$$

and

$$\text{MM}' = d\,(\tan)^{-1}x = \frac{dx}{1+x^2},$$

therefore,

$$\text{area} = \int \frac{y\,dx}{(1+x^2)\sqrt{1+x^2+y^2}}. \qquad (36)$$

§ 15.—*Equation of a spherical Conic in rectangular Coordinates.*

Any equation of the second degree between the spherical coordinates x and y, represents a spherical conic.—(See § 1.) But in order to obtain the equation of a spherical conic in a simple form, we may take its principal diametral arcs for the arcs of reference. The equation of the conic referred to them will be

$$a^2y^2 + b^2x^2 = a^2b^2, \qquad (37)$$

where a and b are the tangents of the distances of the origin from the points in which the curve meets the x and y arcs of reference.

After what has been said in § 1, it is scarcely necessary to give any detailed proof of this. For, if the conic and its diametral arcs be projected by means of radii of the sphere upon a plane touching the sphere at the centre of the conic, the projection of the curve will be an ellipse having for its semiaxes the lines a and b, which are the projections of the principal semidiametral arcs of the spherical conic. Moreover, the equation of the spherical conic is identically the same as that of the ellipse referred to its principal axes. It will be convenient to denote the greatest and least semidiametral arcs, or as I shall henceforth call them, semidiameters of the conic by α and β, so that $a = \tan \alpha$, $b = \tan \beta$. We may define the foci of the conic as two points on the greatest diameter, whose distances from the centre, $+\gamma$ and $-\gamma$, are determined by the equation

$$\cos \gamma = \frac{\cos \alpha}{\cos \beta}. \qquad (38)$$

From this relation we find

$$\frac{\sin^2\gamma}{\sin^2\alpha} = \frac{a^2-b^2}{a^2},\ \frac{\tan^2\gamma}{\tan^2\alpha} = \frac{a^2-b^2}{a^2(1+b^2)},\ \text{and}\ \frac{\sin^2 2\gamma}{\sin^2 2\alpha} = \frac{(a^2-b^2)(1+b^2)}{a^2}. \qquad (39)$$

These three quantities

$$\frac{\sin\gamma}{\sin\alpha}, \frac{\tan\gamma}{\tan\alpha}, \text{ and } \frac{\sin 2\gamma}{\sin 2\alpha},$$

as we shall see presently, will be found to appear in theorems and formulæ relating to spherical conics, in place of the excentricity which we meet in the theory of the plane conic sections. I shall denote them respectively by e, ϵ, and ϵ'.

If in the formulæ already given for the transformation of coordinates (14) we make $\alpha = 90^\circ$, they would become

$$x = -\frac{1}{x'}, \text{ and } y = -\frac{y'}{x'}.$$

Substituting these values of x and y in (37), and removing the accents from x' and y', we should obtain the equation

$$a^2 b^2 x^2 - a^2 y^2 = b^2,$$

which is that of a spherical conic; the y and x arcs of reference being the great circles respectively lying in the principal plane, and in the plane of the greatest section of the cone whose generatrices are the radii drawn to all the points of the conic.

Transferring the origin from the centre to the vertex at the extremity of the greatest diameter, by putting $-a$ for tan α, in formulæ (14), we should find for the equation of the conic

$$y^2 = \frac{2b^2}{a} x - \frac{b^2(1-a^2)}{a^2} x^2. \tag{41}$$

If $a = 1$, that is, if the greatest diameter of the curve becomes a quadrant, the last equation loses the term involving x^2, and the curve becomes what Mr. Davies calls a spherical parabola.

The quantity $\frac{2b^2}{a}$ appears to hold the same place in the theory of the spherical conics that it does in that of the plane conic sections. It may be called the *principal parameter*, and denoted by p.

Transferring the origin from the centre to the focus, by putting $\pm\gamma$ for α in formulæ (14) and afterwards removing the accents from x' and y', we should obtain the equation of a spherical conic in the following form,

$$y^2 + x^2 = (\tfrac{1}{2}p \mp \epsilon' x)^2. \tag{42}$$

Now, this equation represents, not only the spherical conic, but also the plane curve formed by the intersection of the tangent plane at the focus with the cone whose generatrices are the radii of the sphere drawn to all the points of the conic: and it is evident from the form of equation (42) that this plane curve will be a conic section having its focus at the point of contact, its semiparameter equal to p, and its excentricity equal to ϵ'.

Making $x = 0$ in (42), we shall find that the tangent of the arc, drawn from the focus to the curve, and perpendicular to the greatest diameter, is equal to half the principal parameter.

Again, without resorting to the formula (22), we may show that the tangent of the radius of the small circle osculating the conic at the extremity of the greatest diameter, is half the principal parameter. In order to prove this, I must premise the following proposition.

If an arc k be drawn from any point in the circumference of a small circle of the sphere, perpendicular to a fixed diametral arc d of the circle, and dividing it into two segments s and s', we should always have

$$\tan^2 k = \frac{\sin s \sin s'}{\cos^2 \frac{1}{2} d}:$$

and the value of $\frac{\tan^2 k}{2 \sin s'}$, when k and s' are both $= 0$, is evidently $\tan \frac{1}{2} d$.

Hence, if the equation of a spherical curve be given in rectangular coordinates, the arcs of reference being the tangent and normal arcs at a point on the curve, we should obtain the tangent of the radius of the circle of the sphere which osculates the curve at the origin by finding the value of $\frac{y^2}{2x}$, when x and y are each $= 0$.

Applying this principle to the spherical conic, represented by the equation (41), we should find the tangent of the radius of the circle osculating the conic at the origin to be equal to half the principal parameter.

§ 16.—*Polar Equation of a spherical Conic.*

The polar equation of a spherical conic referred to its centre as pole may be obtained from (37) by the method stated in § 14. We thus find it to be

$$\tan^2 \varrho = \frac{\tan^2 \beta}{1 - \epsilon^2 \cos^2 \omega}, \tag{43}$$

which also gives

$$\sin^2 \varrho = \frac{\sin^2 \beta}{1 - \varepsilon^2 \cos^2 \omega}. \tag{44}$$

It appears from (43) that the sum of the squares of the cotangents of two rectangular semidiameters is constant.

Equation (44) shows that the orthographic projection of a spherical conic, upon a plane touching the sphere at the centre of the conic will be an ellipse.

Hence, if arcs PM, PN, be drawn from a point on a spherical conic perpendicular to its greatest and least diameters, we should have

$$\frac{\sin^2 \text{PN}}{\sin^2 \alpha} + \frac{\sin^2 \text{PM}}{\sin^2 \beta} = 1. \qquad (45)$$

It would not, however, be advantageous in general to take the sines of the arcs drawn from a point on the sphere perpendicular to two fixed rectangular arcs as the spherical coordinates of that point.

Using these coordinates, and calling them X and Y, we should find that the equation of a great circle was of the second degree, viz.:

$$\frac{X^2}{\sin^2 \alpha} + \frac{2XY}{\tan \alpha \tan \beta} + \frac{Y^2}{\sin^2 \beta} = 1,$$

where α and β denote the distances of the origin from the points in which the great circle meets the arcs of reference.

The polar equation of a spherical conic referred to its focus as pole is found from (42) to be

$$\tan \rho = \frac{\frac{1}{2} p}{1 \pm \epsilon' \cos \omega}. \qquad (46)$$

We infer from this, that if any arc be drawn through the focus of a spherical conic, meeting the curve in two points, the sum of the trigonometric cotangents of the arcs lying between the focus and these two points will be constant.

It is easy to show that

$$\frac{\tan^2 \beta}{\tan \alpha} = \frac{\cos 2\gamma - \cos 2\alpha}{\sin 2\alpha}.$$

Equation (46) may therefore be put into the following form,

$$\tan \rho = \frac{\cos 2\gamma - \cos 2\alpha}{\sin 2\alpha \pm \sin 2\gamma \cos \omega}. \qquad (47)$$

If we investigate by polar coordinates the locus of a point on the surface of a sphere the sines of whose distances from a fixed point on the sphere and a fixed arc are always in the ratio of m to 1, we should find for the equation of the locus

$$\tan \rho = \frac{m \sin \delta}{1 - m \cos \delta \cos \omega}, \qquad (48)$$

where δ is the distance of the fixed point from the fixed arc.

Comparing this equation with (46) we have

$$m \sin \delta = \tfrac{1}{2} p, \text{ and } m \cos \delta = \epsilon',$$

so that

$$m^2 = \tfrac{1}{4} p^2 + \epsilon'^2, \text{ and } \tan \delta = \frac{p}{2\epsilon'}, \qquad (49)$$

§ 17.—*Equations of the cyclic Arcs of a spherical Conic.*

A spherical conic being formed by the intersection of the sphere with a cone of the second degree, having its vertex at the centre of the sphere, the *cyclic arcs* of this conic are the great circles whose planes are parallel to two subcontrary circular sections of the cone. Consequently, if a plane be drawn touching the sphere at the pole of one of these great circles, it will intersect the cone in a circle. Now, the equation of that circle in the tangent plane is the same as that of the spherical conic, if the point of contact be made the origin of rectangular coordinates in both cases. Further, we may assume that the poles of the cyclic arcs are on the least diameter of the conic. The question of finding the cyclic arcs is, therefore, reduced to this,—to transfer the origin of rectangular spherical coordinates from the centre of the conic to a point in its least diameter, that diameter being retained as the y arc of reference, and the new coordinates being rectangular, so that in the transformed equation x^2 and y^2 may have the same coefficient. In order to effect this we must use the formulæ (15) for the transformation of coordinates, putting ψ in them instead of β, so as to avoid ambiguity, and then determine ψ by the condition that the coefficients of x^2 and y^2 may be equal. We thus obtain

$$\tan^2\psi = \frac{a^2 - b^2}{b^2(1+a^2)}.$$

Hence, we find the equations of the cyclic arcs of a conic given by the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

to be

$$y = \frac{\pm b\sqrt{1+a^2}}{\sqrt{a^2-b^2}}. \tag{50}$$

Denoting by ϕ the angle between one of the cyclic arcs and the greatest diameter, we get from this, since $\phi + \psi = 90^\circ$,

$$\cos\phi = \frac{\tan\gamma}{\tan\alpha}, \text{ and } \sin\phi = \frac{\sin\beta}{\sin\alpha}. \tag{51}$$

These equations show that if two spherical conics have the same cyclic arcs, the quantity ι, in their polar equations (44) referred to the centre as pole, will be the same for both: consequently, *the ratio of the sines of those semidiameters of two biconcyclic conics which make equal angles with the greatest diametral arc will be constant: and hence, if two biconcyclic conics be projected orthographically upon a plane touching the sphere at their common centre, the projections will be two similar, similarly placed, and concentric ellipses.*

§ 18.—*Equation of the great Circle touching a spherical Conic at a given Point x', y'.*

The equation of the spherical conic referred to its principal diameters being

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

we find from (16) the equation of the tangent arc at the point x', y', to be

$$a^2 y' y + b^2 x' x = a^2 b^2. \qquad (52)$$

Hence, also, we may prove, in the same way as the similar result is arrived at in the theory of the plane conic sections, that if tangent arcs be drawn to a conic from a point without it, x', y', the equation of the arc joining the points of contact will be

$$a^2 y' y + b^2 x' x = a^2 b^2. \qquad (53)$$

Since x and y may be interchanged with x' and y', without altering the form of the equation (53), it must also represent the locus of the point of concourse of the two tangent arcs drawn to a conic at the points where any arc passing through a fixed point x', y', meets the curve.

The arc touching the conic at the point x', y', meets the principal diameters of the conic in points, the tangents of whose distances from the centre are $\frac{a^2}{x'}$ and $\frac{b^2}{y'}$: call these X and Y, and we shall have, since $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$X^2 b^2 + Y^2 a^2 = X^2 Y^2.$$

This shows that the spherical conic, the tangents of whose principal semidiameters are X and Y, will pass through the point whose coordinates are a and b.

§ 19.—*Conjugate Diameters of a spherical Conic.*

Conjugate diameters of a spherical conic are those each of which passes through the pole of the other with relation to the conic. The poles of both of them are evidently on the great circle whose pole is the centre of the conic.

From what has been said in § 1, with reference to the meaning of the constants in the equation of an arc of a great circle, it appears that the diameter whose equation is

$$a^2 y' y + b^2 x' x = 0, \qquad (54)$$

is conjugate to the diameter drawn to the point x', y', and whose equation is therefore,

$$x' y - y' x = 0:$$

for it meets the great circle whose pole is the centre of the conic in the same point that the tangent arc at the point x', y', meets that same circle.

Any two conjugate diameters of a spherical conic will evidently make with the greatest diameter two angles the product of whose tangents is constant and equal to $-\frac{b^2}{a^2}$.

The diameter conjugate to that which passes through the point x', y', on the conic, will meet the curve in two points whose coordinates x'', y'', are found from the equations (54) and (37) to be

$$x'' = \pm \frac{a}{b} y', \text{ and } y'' = \pm \frac{b}{a} x'. \qquad (55)$$

Let us denote the semidiameter drawn to the point x', y', by r', and the semiconjugate diameter by r'', then as $\tan^2 r'' = x''^2 + y''^2$, we shall have

$$\tan^2 r'' = \frac{a^4 y'^2 + b^4 x'^2}{a^2 b^2}. \qquad (56)$$

The equation of the tangent arc at that extremity of the conjugate diameter, for which the coordinates are

$$x'' = -\frac{a}{b} y', \text{ and } y'' = \frac{b}{a} x', \text{ is } x' y - y' x = ab. \qquad (57)$$

It may be observed that since

$$x'^2 + x''^2 = a^2, \text{ and } y'^2 + y''^2 = b^2,$$

the sum of the squares of the tangents of two conjugate semidiameters of a spherical conic is constant.

§ 20.—*To determine the lengths of the Arcs drawn from the Centre or Focus perpendicular to Tangent Arc at the Point x', y'.*

Let λ be the arc drawn from the centre, perpendicular to the tangent arc at the point x', y': by the aid of formulæ (9) and (52), we find

$$\left.\begin{aligned} \sin \lambda &= \frac{a^2 b^2}{\sqrt{a^4 b^4 + a^4 y'^2 + b^4 x'^2}}, \\ \text{and} \quad \tan \lambda &= \frac{a^2 b^2}{\sqrt{a^4 y'^2 + b^4 x'^2}}. \end{aligned}\right\} (58)$$

The last formula compared with (56) shows that $\tan r'' \tan \lambda = ab$: that is

In a spherical conic, the product of the trigonometric tangents of the perpendicular let fall from the centre of the conic on a tangent arc, and of the semidiameter conjugate to that which passes through the point of contact, is constant.

Let l be the arc drawn from the focus, whose coordinate is $+\tan\gamma$, $(= a\iota,)$ perpendicular to the tangent arc at the point x', y'; and let l' be the arc drawn from the other focus, whose coordinate is $-a\iota$, perpendicular to the same tangent arc. Then, from formulæ (9) and (52), we have

$$\sin l = \frac{ab^2(a-\iota x')}{\sqrt{(a^4b^4+a^4y'^2+b^4x'^2)(1+a^2\iota^2)}}, \qquad (59)$$

and

$$\sin l' = \frac{ab^2(a+\iota x')}{\sqrt{(a^4b^4+a^4y'^2+b^4x'^2)(1+a^2\iota^2)}}. \qquad (60)$$

Now, the quantity $a^4b^4+a^4y'^2+b^4x'^2$ may be put into the form $a^2b^2(1+b^2)(a^2-\iota^2x'^2)$: consequently,

$$\sin l \sin l' = \frac{b^2}{1+a^2}. \qquad (61)$$

Therefore, in any spherical conic, the product of the sines of the arcs drawn from the two foci perpendicular to any tangent arc is constant.

§ 21.—*A spherical Conic being represented by the Equation* $a^2y^2+b^2x^2=a^2b^2$, *to express the Distances* ρ', ρ'', *of a Point* x', y', *on the Curve from the two Foci.*

To accomplish this we have only to put $\pm a\iota$ and 0, in place of x'' and y'', in formula (8): we thus find

$$\tan^2\rho' = \frac{(x'-a\iota)^2+y'^2(1+a^2\iota^2)}{(1+a\iota x')^2}. \qquad (62)$$

Now the quantity $(x'-a\iota)^2+y'^2(1+a^2\iota^2)$ may be put into the form $(a-\iota x')^2$; we have, therefore,

$$\tan\rho' = \frac{a-\iota x'}{1+a\iota x'}; \qquad (63)$$

and in like manner

$$\tan\rho'' = \frac{a+\iota x'}{1-a\iota x'}. \qquad (64)$$

From the form of these two last equations it is plain that

$$\rho' = \alpha - (\tan)^{-1}\iota x', \text{ and } \rho'' = \alpha + (\tan)^{1-}\iota x';$$

hence

$$\rho' + \rho'' = 2\alpha. \qquad (65)$$

Therefore, the sum of the distances of any point on a spherical conic from the two foci is constant.

It is to be observed, that the tangent of the distance of any point x', y', on a spherical conic from either focus is a rational function of x'.

Had we employed formula (7) instead of (8) we should have found the following values of $\sin\varrho'$ and $\sin\varrho''$, which we shall have occasion presently to refer to:

$$\sin\varrho' = \frac{a - \iota x'}{\sqrt{(1 + a^2\iota^2)(1 + x'^2 + y'^2)}}. \qquad (66)$$

$$\sin\varrho'' = \frac{a + \iota x'}{\sqrt{(1 + a^2\iota^2)(1 + x'^2 + y'^2)}}. \qquad (67)$$

Since

$$\frac{\sin l}{\sin\varrho'} = \frac{\sin l'}{\sin\varrho''},$$

if follows that the arcs drawn from the two foci to any point on the curve make equal angles with the tangent arc at that point.

Let θ denote the angle between either of these vector arcs and the tangent arc; then we shall have

$$\sin^2\theta = \frac{\sin l \sin l'}{\sin\varrho' \sin\varrho''}.$$

Hence

$$\sin^2\theta = \frac{a^2 b^4 (1 + x'^2 + y'^2)}{a^4 b^4 + a^4 y'^2 + b^4 x'^2}. \qquad (68)$$

§ 22.—*Equation of the normal Arc of a spherical Conic.*

We find from equations (12) and (53) that the equation of the arc passing through the point x', y', and perpendicular to the arc which joins the points of contact of the two tangent arcs drawn to the conic from the point x', y', is

$$\frac{y - y'}{x - x'} = \frac{y' a^2 (1 + b^2)}{x' b^2 (1 + a^2)} = \frac{y' \sin^2\alpha}{x' \sin^2\beta}. \qquad (69)$$

If the point x', y', be on the curve, this same equation becomes that of the normal arc at the point.

By making y and x successively $= 0$ in (69), we find the coordinates x'' and y'', of the points in which the arc represented by equation (69) meets the greatest and least diameters to be

$$x'' = \frac{a^2 - b^2}{a^2 (1 + b^2)} x', \text{ and } y'' = \frac{b^2 - a^2}{b^2 (1 + a^2)} y'. \qquad (70)$$

But the arc which joins the points of contact of the two tangent arcs drawn to the conic from the point x', y', meets the greatest and least diameters in points whose coordinates x_1 and y_1, are given by the equations

$$x_1 = \frac{a^2}{x'}, \text{ and } y_1 = \frac{b^2}{y'}.$$

comparing these values with those of x'' and y'', given in (70), we have

$$\left.\begin{aligned} x'' x_1 &= \frac{a^2 - b^2}{1 + b^2} = \tan^2\gamma, \\ y'' y_1 &= \frac{b^2 - a^2}{1 + a^2} = -\sin^2\gamma. \end{aligned}\right\} \quad (71)$$

These last equations show, that if two tangent arcs be drawn from any point to a spherical conic, and another arc be drawn from the same point perpendicular to the arc which joins the points of contact, these two arcs, which are at right angles to each other, will meet either of the principal diameters of the conic in two points, the product of the coordinates of which is constant.

Hence, if two tangent arcs be drawn to a spherical conic, and if the arc joining the points of contact touch a second spherical conic which has the same foci as the first, the arc joining the point of concourse of the tangents to the first conic with the point of contact on the second curve, will be a normal to the latter.

As a particular case of the last theorem we may deduce the following:

Any arc passing through the focus of a spherical conic is perpendicular to the arc joining that focus with the point of concourse of the tangent arcs drawn to the conic at the two points in which the arc passing through the focus meets the curve.

It appears from (51), that equation (69) is the same for all spherical conics which have the same cyclic arcs. Hence, if tangent arcs be drawn to one of two biconcyclic conics from a point on the other, the arc drawn from that point perpendicular to the arc joining the points of contact on the first conic, will be a normal to the other curve.

Equations (70) show that if any number of spherical conics, which have the same cyclic arcs, be cut by an arc perpendicular to either of the principal diameters common to all the curves, the normals at the points in which this arc meets the several conics will all pass through the same point on that principal diameter.

In order to find the length of the normal arc ν, drawn from the point x', y', to meet the greatest diameter we must put in (8) $y'' = 0$, and $x'' = \iota^2 x'$; (see 70): thus we obtain

$$\tan^2\nu = \frac{x'^2 (1 - \iota^2)^2 + y'^2 (1 + \iota^4 x'^2)}{(1 + \iota^2 x'^2)^2};$$

to this expression we may give an elegant and symmetrical form ; for

$$1-\iota^2=\frac{b^2}{a^2}(1+a^2\iota^2),\ \text{and}\ y'^2=\frac{b^2}{a^2}(a^2\quad x'^2);$$

the numerator in the value of $\tan^2\nu$ may, therefore, be written thus,

$$\frac{b^2}{a^2}[x'^2(1-\iota^2)(1+a^2\iota^2)+(a^2-x'^2)(1+\iota^4x'^2)],$$

which is evidently equal to

$$\frac{b^2}{a^2}(a^2-\iota^2x'^2)(1+\iota^2x'^2).$$

Expunging the common factor $1+\iota^2x'^2$, and observing that

$$a^4b^4+a^4y'^2+b^4x'^2=a^2b^2(1+b^2)(a^2-\iota^2x'^2),$$

and

$$1+x'^2+y'^2=(1+b^2)(1+\iota^2x'^2), \tag{72}$$

we obtain finally

$$\tan^2\nu=\frac{a^4b^4+a^4y'^2+b^4x'^2}{a^4(1+x'^2+y'^2)}. \tag{73}$$

In like manner we find the length of the normal arc ν', drawn from the point x', y', to meet the least diameter.

$$\tan^2\nu'=\frac{a^4b^4+a^4y'^2+b^4x'^2}{b^4(1+x'^2+y'^2)}. \tag{74}$$

It is evident that

$$\frac{\tan\nu}{\tan\nu'}=\frac{b^2}{a^2}.$$

From equations (68), (73), and (74) we have

$$\tan\nu\sin\theta=\tfrac{1}{2}p,\ \text{and}\ \tan\nu'\sin\theta=a.$$

Hence, since the angle between the normal and either of the focal vector arcs drawn to the point x', y', is the complement of θ, we derive the following theorem.

From the points where the normal to a spherical conic meets its greatest and least diameters, if arcs be drawn perpendicular to either of the focal vector arcs passing through the point on the curve at which the normal is drawn, these arcs will cut off constant portions from the vector arc.

Comparing equations (66) and (67) with (73) and (74) we find

$$\tan \nu \tan \nu' = \sin \varrho \sin \varrho' (1 + a^2). \quad (75)$$

Let us denote by ζ the angle between the normal and the arc drawn from the centre to the point x', y', then we shall have

$$\sin \lambda = \sin r' \cos \zeta, \text{ and } \sin \lambda \tan \nu = b^2 \cos r';$$

therefore,

$$\tan \nu \tan r' \cos \zeta = b^2. \quad (76)$$

Consequently, if we denote by ξ the portion cut off from the normal by a perpendicular let fall upon it from the centre, we obtain

$$\tan \nu \tan \xi = b^2. \quad (77)$$

§ 23.—*Equations of the director Arcs of a spherical Conic.*
The director arc of a spherical conic is the locus of the point of concourse of the two tangent arcs drawn to the conic at the points where any arc passing through the focus meets the curve.

The equations of the director arcs corresponding to the two foci are found from (53) to be

$$x = \pm \frac{a}{\iota}. \quad (78)$$

In formula (9), let us make

$$\beta = 0, \text{ and } \alpha = \pm \frac{\iota}{a},$$

and it will give for the value of the sine of the perpendicular κ let fall from the point x', y', on the director arc

$$\sin \kappa = \frac{a \mp \iota x'}{\sqrt{(a^2 + \iota^2)(1 + x'^2 + y'^2)}}. \quad (79)$$

Comparing this equation with (66) and (67), we see that the sines of the distances of any point on the conic from a focus and the corresponding director arc are to each other in a constant ratio. If m be the exponent of this ratio,

$$m^2 = \frac{a^2 + \iota^2}{1 + a^2 \iota^2}. \quad (80)$$

§ 24.—*Equation of the spherical Conic supplementary to the one represented by the Equation*

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

The spherical curve *supplementary* to a given one is the locus of the poles of all the great circles which touch the given curve.

The equation of the tangent arc at the point x', y', to the conic represented by the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, has been shown to be

$$a^2 y' y + b^2 x' x = a^2 b^2.$$

The coordinates of the pole of this circle x'' and y'', are by (2)

$$x'' = -\frac{x'}{a^2}, \text{ and } y'' = -\frac{y'}{c^2}. \tag{81}$$

Hence, putting the values of x' and y' found from these last equations in $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, and removing the accents from x'' and y'', as no longer necessary, we find the equation of the proposed locus to be

$$b^2 y^2 + a^2 x^2 = 1; \tag{82}$$

as the equation of the given conic may be written in the form

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1,$$

it is plainly supplementary to the conic represented by the equation (82).

Since $a > b$, the foci of the supplementary conic are on the least diameter of the given one. Let us denote the greatest and least semidiameters of the supplementary conic by α' and β', and the distance between its foci by $2\gamma'$; then we shall plainly have

$$\alpha' + \beta = \alpha + \beta' = 90^\circ,$$

so that

$$\cos \gamma' = \frac{\cos \alpha'}{\cos \beta'} = \frac{\sin \beta}{\sin \alpha} = \sin \phi.$$

Therefore, the *foci* of the supplementary conic are the poles of the cyclic arcs of the given conic: and, since the curves are mutually supplementary, the foci of the given conic are the poles of the cyclic arcs of the supplementary one.

But we may prove the existence of a more general relation between the two curves in the following manner:

The equation of the locus of the point of concourse of tangents drawn to the given conic at the points in which any arc passing through the point x', y', meets the curve, being

$$a^2 y' y + b^2 x' x = a^2 b^2;$$

the coordinates of the pole of this great circle are

$$-\frac{x'}{a^2}, \text{ and } -\frac{y'}{b^2}.$$

Again, the equation of the locus of the point of concourse of tangents drawn to the supplementary conic at the points in which any arc passing through the point whose coordinates are

$$-\frac{x'}{a^2}, \text{ and } -\frac{y'}{b^2},$$

meets the curve, will be

$$x'x + y'y = -1;$$

and the coordinates of the pole of this great circle are x' and y'.

Thus, we have proved that to a point and its polar arc, with relation to a given spherical conic, correspond an arc and its pole with relation to the supplementary conic. It appears from this that to a focus and its director arc in a given conic, correspond a cyclic arc in the supplementary conic, and its pole with relation to that curve.

The two supplementary conics being thus connected, we are able, when a theorem has been proved with relation to points and arcs belonging to a spherical conic, to deduce from it at once another theorem relating to the corresponding arcs and points belonging to the supplementary conic.

§ 25.—*Evolute and osculating Circle of a spherical Conic.*
By differentiating the equation (37) twice we should find

$$y'dx' - x'dy' = \frac{b^2\,dx'}{y'}.$$

$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2 = \frac{a^4 b^4 + a^4 y'^2 + b^4 x'^2}{a^4 y'^2}\,dx'^2. \quad (83)$$

$$dx' + y'(y'dx' - x'dy') = (1 + b^2)\,dx'.$$

$$dy' + x'(x'dy' - y'dx') = -\frac{b^2 x'}{a^2 y'}(1 + a^2)\,dx'.$$

$$dy'\,d^2x' - dx'\,d^2y' = \frac{b^4\,dx'^3}{a^2 y'^3}. \quad (84)$$

Making these substitutions in formulæ (20), and observing that

$$a^4 b^4 + a^4 y'^2 + b^4 x'^2 = a^2 b^2 (1 + b^2)(a^2 - \epsilon^2 x'^2),$$

and

$$(1 + x'^2 + y'^2) = (1 + b^2)(1 + \epsilon^2 x'^2).$$

We obtain finally the coordinates of the centre of the osculating circle

$$x = \frac{a^2 - b^2}{a^4(1 + b^2)} x'^3, \text{ and } y = \frac{b^2 - a^2}{b^4(1 + a^2)} y'^3.$$

So that the equation of the evolute will be

$$\left(\frac{x}{a'}\right)^{\frac{2}{3}} + \left(\frac{y}{b'}\right)^{\frac{2}{3}} = 1,$$

where

$$a' = \frac{a^2 - b^2}{a(1 + b^2)}, \text{ and } b' = \frac{b^2 - a^2}{b(1 + a^2)}.$$

Substituting from (83) and (84) in formula (22), we get for the semidiameter of the osculating circle

$$\tan \gamma = \frac{(a^4 b^4 + a^4 y'^2 + b^4 x'^2)^{\frac{3}{2}}}{a^4 b^4 (1 + x'^2 + y'^2)^{\frac{3}{2}}} = \frac{\tan^3 \nu}{\frac{1}{4}p^2}.$$

§ 26.—*Rectification and Quadrature of a spherical Conic.*

The formula already given (24) for the differential of the arc of a spherical curve becomes, when we put in their values (83) and (72) for

$$dx'^2 + dy'^2 + (x'dy' - y'dx')^2,$$

and

$$1 + x'^2 + y'^2,$$

$$ds = \frac{bdx \sqrt{a^2 - \iota^2 x^2}}{a\sqrt{1 + b^2}(1 + \iota^2 x^2) y}.$$

Let us make $x = a \sin \varphi$, then, by the equation of the conic (37), $y = b \cos \varphi$; and the differential may be brought into the following form

$$ds = \frac{1}{a\sqrt{1 + b^2}} \left\{ \frac{1 + a^2}{1 + a^2 \iota^2 \sin^2\varphi} - 1 \right\} \frac{d\varphi}{\sqrt{1 - \iota^2 \sin^2\varphi}}.$$

The arc is, therefore, represented by means of two elliptic integrals of the first and third orders, having ι for their common modulus; the parameter of the latter being $a^2 \iota^2$.

To find the area of the conic we may employ the formula (36). In it let us make

$$x = a \cos \theta, \text{ and } y = b \sin \theta,$$

as the equation of the conic permits, then we shall have

$$d\,(\text{area}) = \frac{b}{a\sqrt{1+a^2}}\left\{1 - \frac{1}{1-\sin^2\alpha\,\sin^2\theta}\right\}\frac{d\theta}{\sqrt{1-\sin^2\gamma\,\sin^2\theta}}.$$

The area may, therefore, be made to depend upon two elliptic integrals of the first and third orders ; the parameter of the latter being $-\sin^2\alpha$, and their common modulus $\sin\gamma$.

It may be observed, that the elliptic integral of the third order which presents itself in the quadrature of a spherical conic is of the *circular* and not of the *logarithmic* kind, since $\sin^2\alpha$ is intermediate between $\sin^2\gamma$ and 1.

THE END.